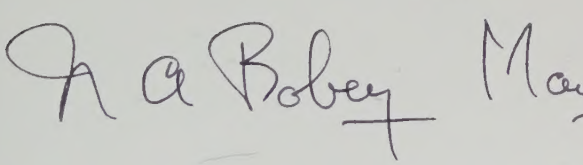
Notes on the Mathematics of The Principles of Electricity



Digitized by the Internet Archive
in 2025

https://archive.org/details/bwb_P9-EBO-372

Notes on the Mathematics of The Principles of Electricity

Compiled for Use with Training Courses
Conducted for Employees of the

Long Lines Department

American Telephone and Telegraph Company

January, 1930

PREFACE

These Notes are designed to serve as supplementary text for use when needed in connection with Training Courses conducted in the Long Lines Department. Their aim is to present only such elementary mathematical concepts and rules as may be required for a reasonably complete understanding of the more fundamental technical problems that will be encountered in this Company's Training Course texts.

Consistent with this purpose, an attempt has been made to treat each branch of mathematics taken up in the briefest and simplest manner possible. Any phases of a subject not considered absolutely essential to the limited objective are omitted, only those particular concepts are treated which it is definitely known the student will need to apply. In the treatment of Algebra, for instance, the sole purpose is to acquaint the reader with methods of handling and solving simple first degree equations. More advanced parts of the subject such as quadratic equations, determinants, etc., are omitted as are several of the more fundamental parts such as fractions and proportion.

It is probable that many readers will be familiar with Algebra from previous studies. To these it is suggested that their study of these Notes begin with Section IV which deals with simultaneous equations. This section will serve as a review of Algebra and as an introduction to the three final sections covering Logarithms, Trigonometry and Vector Algebra, the subjects which it is assumed the majority of readers will be most interested in.

Finally, it is to be emphasized again, that these Notes are in no sense to be considered as a complete treatise on any of the mathematical subjects covered. Indeed, there would have been no necessity for compiling them were this the case, since there are available many excellent texts dealing comprehensively with each subject. The study of one or more of these standard texts is recommended to the student who wishes to obtain a more complete knowledge of mathematics. For the student whose more immediate interest, however, lies in following the exposition in some Training Course text that may call for some mathematical background and who does not wish to digress far from the subject at hand it is hoped that these Notes will be of real service and a stimulus to further study.

CONTENTS

Page

I. **Algebra—Symbols and Signs** 1
1. General—2. Algebraic symbols—2a. Number symbols—2b. Operation symbols—2c. Relation symbols—2d. Aggregation symbols—3. Positive and negative quantities—4. Determination of signs in the fundamental operations—4a. Addition—4b. Subtraction—4c. Multiplication—4d. Division—5. Examples.

II. **Algebra—Addition, Subtraction, Multiplication, Division** 15
6. Addition and subtraction—7. Removal of parentheses—8. Multiplication—9. Division—10. Examples.

III. **Algebra—Simple Equations** 28
11. Definitions—12. Properties of equations—13. Algebraic transformations—14. Solution of simple equations—15. Examples.

IV. **Algebra—Simultaneous Equations** 38
16. Equations containing more than one unknown quantity—17. Methods of solving simultaneous equations—17a. Elimination by addition or subtraction—17b. Elimination by substitution—17c. Elimination by comparison—18. Solution of equations containing more than two unknowns—19. Examples.

V. **Logarithms** 53
20. Properties of exponents—21. Definition of logarithm—22. Logarithmic tables—23. Uses of logarithms—24. Examples.

VI. **Elements of Trigonometry** 68
25. Review of plane geometry—25a. The circle 25b. Angular measure—25c. The right triangle 26. Trigonometric functions—27. Functions of angles greater than 90°—28. Trigonometric formulae—29. Examples.

VII. **Vector Algebra** 92
30. Vector quantities—31. Vector notation—32. Operations on vectors—33. Calculations with vector quantities—34. Examples.

Table of Common Logarithms 60

Table of Trigonometric Functions 79

I. ALGEBRA—Symbols and Signs

1. General

Algebra may be thought of as a generalization of Arithmetic. Or, viewed from another angle, Arithmetic is a branch of Algebra that deals exclusively with numbers; Algebra is the more fundamental mathematical science. It deals with numbers and with symbols representing known or unknown numbers or quantities. This broadens its scope considerably as compared with Arithmetic and makes it an extremely valuable tool in solving many types of problems. Its notation is also admirably adapted to expressing briefly exact physical laws or relationships in the form of equations or formulas.

2. Algebraic Symbols

In the study of Algebra, as in that of any other science, we must first learn the conventional terminology. We must become familiar with the significance of the symbolic notation which is the special language of the subject. Fortunately, this is not difficult.

The symbols representing number or quantity in Arithmetic are the well known Arabic numerals. These symbols are used for the same purpose in Algebra but also the letters of the alphabet are used likewise to represent numbers or quantities. The numerals can obviously be used only to represent definitely known quantities, but letters may denote quantities either known or unknown.

2a. Number Symbols

In purely mathematical expressions it is customary to use the letters of the end of the alphabet, as

x, y and z, to represent unknown quantities, and letters at the beginning of the alphabet, as a, b, c, etc., to denote known quantities. In equations or expressions involving physical quantities, however, the symbols used are selected to permanently represent each particular kind of physical quantity quite independently of whether the magnitude or value of the quantity is known or unknown. In many cases such symbols are in the nature of abbreviations, the physical quantity being represented by the first letter of its name, as R for resistance, l for length, t for time, etc. In other cases entirely arbitrary symbols must be used to avoid confusion. For this purpose, the letters of the Greek alphabet are commonly employed, as, for instance, α for attenuation, μ for permeability, Θ for the measure of an angle, and so on. This makes necessary a method of differentiating between different numbers or amounts of the same physical quantity occurring in a single expression or group of related expressions. Thus the symbol, I, is customarily used to represent electrical current and if several currents of different values are to be represented in a single expression, there must be some way provided for distinguishing the several different I's. There are two standard conventions for doing this. The first makes use of **accents**, as I′, I″, I‴ (read I prime, I second, I third) etc.; the second employs numerical **subscripts**, as I_1, I_2, I_3 (read I one, I two, I three) etc.

2b. Operation Symbols

The symbols for the operations of **addition, subtraction, multiplication** and **division**, $+$, $-$, $\times$ and $\div$, respectively, have exactly the same significance

in Algebra as in Arithmetic. The sign of multiplication, $\times$, is usually omitted in Algebra, however, due to its resemblance in appearance to the letter x. Thus, if we wish to indicate the product of a and b or of I and R, we write simply ab or IR instead of a $\times$ b or I $\times$ R. The operation of division is perhaps more frequently indicated by the fractional form in Algebra than in Arithmetic. That is, instead of writing $x \div y$ or E $\div$ I, we usually express the same thing in the form x/y or E/I.

A **power** of a quantity is the result of multiplying the quantity by itself a given number of times. The number of times that the quantity is used in obtaining the product or power is indicated by a number written to the right of and above the quantity and called the **exponent.**

Thus—

a^2 denotes $a \times a$ or aa and is called "a square" or "a to the second power";

a^3 denotes $a \times a \times a$ or aaa and is called "a cube" or "a to the third power";

3^4 means $3 \times 3 \times 3 \times 3$ or 81; and so on.

A **root** of a number is a quantity which when multiplied by itself a given number of times will produce the number. Thus the "third" or "cube" root of a^3 is a because $a \times a \times a$ equals a^3; similarly, 8 is the "second" or "square" root of 64 because 8×8 equals 64. The operation of finding the root of a number is called **evolution** and is indicated by writing the **radical** sign, $\surd$, before the number. A small number written to the left of and above the radical sign is called the **index** and indicates which

root is to be taken. For example, the second or square root of x^8 is represented by $\sqrt[2]{x^8}$ or, more commonly, $\sqrt{x^8}$, the value of the index number being understood as 2 when omitted; similarly, $\sqrt[3]{64}$ represents the third or cube root of 64, $\sqrt[4]{x^4}$ indicates the fourth root of x^4, and so on.

The **reciprocal** of a number is 1 divided by the number. Thus the reciprocal of x is $1/x$ and the reciprocal of $1/x$ is $1/1/x$ which equals x.

2c. Relation Symbols

The symbols for equality and inequality are the same in Algebra as in Arithmetic.

Thus $x = y$ signifies that the quantities x and y are equal;

$x \neq y$ signifies that x and y are not equal;

$x > y$ means that x is greater in value than y;

and $x < y$ indicates that x is less than y.

2d. Aggregation Symbols

The signs of aggregation or grouping are used when it is desired to indicate that all of the quantities included within them are to be subjected to the same operation. These signs include the parenthesis ( ), the brackets [ ], the braces { }, and the vinculum ————. The last is ordinarily used only in conjunction with the radical sign where it indicates that the root of all of the quantities beneath it is to be taken. Thus, the expression $\sqrt{a + b + c}$ means that the square root of the sum of the quantities a, b and c is to be taken.

Parentheses, brackets and braces all have exactly the same significance, namely that the numbers or quantities enclosed by them are to be treated collectively. $2\ (x + y)$, $2\ [x + y]$, and $2\ \{x + y\}$ all mean that the sum of x and y is to be multiplied by 2. The need for several kinds of signs of aggregation arises when one group of quantities is included within a second group or when this second group is included within still another group, etc.

As an example, consider the expression—

$$\{[(a + b) \div c] \times 8\} \div 2$$

This means that the sum of a and b is to be divided by c, the resulting quotient multiplied by 8 and this product divided by 2. It will be noted that the several operations must be performed in order **starting with the innermost group.**

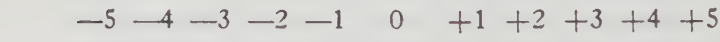
3. Positive and Negative Quantities

The signs + and — are used in both Algebra and Arithmetic to indicate addition and subtraction, respectively. In Algebra they are used also to indicate opposite states of quantities. That is to say, in Algebra we deal with both positive and negative quantities and indicate the state of the quantity by means of the symbols, + and —. When used in this sense, these signs are called **quality** or qualifying signs and have a different significance than when used as symbols of operation alone, (i.e., to indicate addition and subtraction).

The concept of a negative quantity comes from the fact that in many cases physical quantities exist in two opposite states. Thus we may have transmission **losses** and **gains,** temperature **above** zero

and **below** zero, acceleration when a train **speeds up** and an opposite kind of acceleration when it **slows down.** In each case the magnitudes, while a measure of the same kind of quantity, are **in opposite directions.** It is logical from a physical and convenient from a mathematical standpoint to represent these opposites by + and — signs. Thus if we call temperature above zero positive, temperature below zero will be negative; and if we represent a certain transmission gain as + 10 db, then an equal transmission loss must be shown as —10 db.

We may consider, therefore, that while in Arithmetic we count only from zero upwards as 0, 1, 2, 3, 4, 5, 6, etc., in Algebra we count from zero both upwards and downwards. We can represent this on a horizontal scale where values to the right of zero are positive and values to the left are negative, thus—



From this we can see that two numbers of opposite sign such, for example, as +3 and —3 are radically different although both have the same **absolute** value, namely 3. In Algebra a number is not defined unless both its absolute value and its sign are given. It is customary, however, to omit the + sign; when a number is written simply 2 or x, it is to be understood as +2 or + x.

Either a positive or a negative quantity may be represented by a letter symbol. We may, for example, let the letter n stand for +4 or —4 or any other number on our scale, positive or negative.

The negative symbol, $-n$, will then represent a number of the same absolute value but of opposite sign. That is, if we let n stand for $+4$, $-n$ will represent -4; and if n is selected to represent -4, $-n$ will stand for $+4$, etc. Hence we see that

$$-(+4) = -4$$

and

$$-(-4) = +4$$

It is evident also that

$$+(-4) = -4$$

and

$$+(+4) = +4$$

since placing a plus sign before the symbol has by definition no effect on its value.

From the above we may generalize as follows:

1. **The negative of the positive of a number and the positive of the negative of a number are both equal to the negative of the number.**

2. **The positive of the positive of a number and the negative of the negative of a number are both equal to the positive of the number.**

Both rules may be most easily remembered from the less precise but much briefer statement that—

Like signs produce $+$, and unlike signs produce $-$.

4. Determination of Signs in the Fundamental Operations

4a. Addition

Addition in Algebra is the process of finding the **combined value** of the quantities to be added. As long as the quantities to be added are arithmetical

(i.e., positive), it is identical with the process of addition in Arithmetic. For example, if a boy has \$5.00 in one pocket and \$3.00 in another, his total capital is \$5.00 + \$3.00 = \$8.00 Or, if we let a stand for \$1.00, we can write for his cash on hand—

$$5a + 3a = 8a.$$

Similarly, we may say in general that—

$$a + a = 2a$$

$$6a + 4a = 10a$$

and so on, where a denotes any quantity or number.

But we deal also with negative quantities in Algebra. Such quantities may be added together or added to positive quantities. Assume that in the above example, our boy who has \$8.00 on hand, owed \$4.00 to his father. We know that his net capital would then be \$8.00 — \$4.00 or \$4.00. Using algebraic notation we could say that his capital consisted of two items, +\$8.00 and —\$4.00, or, writing a in place of \$1.00 again, + $8a$ and —$4a$. Adding these, we have

$$(+8a) + (-4a) = (+4a)$$

and, from the preceding Article, this may be written

$$8a - 4a = 4a$$

Similarly, if his cash on hand was \$6.00 and he owed \$11.00, his net capital would be a debt of \$5.00. Using the same terminology as above, we have

$$(+6a) + (-11a) = (-5a)$$

or

$$6a - 11a = -5a$$

Again, suppose that he owes \$3.00 to one friend and \$4.00 to another and has no funds. His net capital is then a debt of \$7.00. We may express this—

$$(-3a) + (-4a) = (-7a)$$

or

$$-3a - 4a = -7a$$

From the foregoing examples we may derive the two following general rules of signs for addition in Algebra:

1. **To add two quantities having the same sign (i.e., both positive or both negative), add their absolute values and prefix the common sign of the two quantities to the result.**
2. **To add two quantities having unlike signs, subtract the smaller absolute value from the greater and prefix the sign of the quantity having the greater absolute value to the result.**

4b. Subtraction

Subtraction in Algebra and Arithmetic are identical processes so long as all of the algebraic quantities are positive. Thus if a man has \$6.00 and spends \$2.00, he has left \$4.00. Considering these quantities all positive, as we normally would, and writing a for \$1.00, we have—

$$+6a - (+2a) = +4a$$

or simply

$$6a - 2a = 4a$$

Similarly

$$2a - (+2a) = 2a - 2a = 0,$$

$$9a - 7a = 2a, \text{ etc.}$$

Now consider this example. A boy has \$14.00 in cash but owes his father \$4.00. We may say that he has assets or a positive capital of \$14.00, which we can represent as +\$14.00, and a debt or negative capital of \$4.00, which we may represent as —\$4.00. His net capital is then +\$10.00.

His father now tells him that he is going to cancel the debt; in other words, take away from the boy's net capital of \$10.00 his negative capital of —\$4.00 The boy's net capital after this procedure is, of course, \$14.00. We may say then, that

$$(+\$10.00) - (-\$4.00) = (+\$14.00)$$

or, representing \$1.00 by a as above,

$$10a - (-4a) = 14a = 10a + 4a$$

Similarly,
$$8a - (-12a) = 8a + 12a = 20a$$
$$2a - (-2a) = 2a + 2a = 4a$$
$$-2a - (-2a) = -2a + 2a = 0$$
$$-6a - (-3a) = -6a + 3a = -3a, \text{ etc.}$$

From a consideration of the above examples, we may deduce a general rule for subtraction as follows:

To subtract one number from another, change the sign of the number to be subtracted and add the two numbers.

4c. Multiplication

In Arithmetic, to multiply one number by another is to add together as many duplicates of the first number as there are units in the second number. Thus

$$4 \times 8 = 8+8+8+8 = 32$$

Similarly, in Algebra

$$4 \times a \text{ or } 4a = a+a+a+a$$

Likewise, $4\times(-a)$ means that four $(-a)$'s are to be added together; that is

$$4\times(-a) = (-a)+(-a)+(-a)+(-a)$$

$$= -a -a -a -a = -4a$$

or, in general, if b is any arithmetical (i.e., positive) number, then

$$b \times (-a) = -ba$$

and

$$b \times (+a) = +ba$$

Now, we know from Arithmetic that the order in which two numbers are multiplied, that is, which is the multiplier and which the multiplicand, is immaterial in so far as the magnitude of the product is concerned. That is to say, 4×5 is the same thing as 5×4, the product in each case being 20. Therefore

$$2\times(-3)=(-3)+(-3)=-6=(-3)\times2$$

and this is also the same in value as

$$3\times(-2)=(-2)+(-2)+(-2)=-6=(-2)\times3$$

It follows from this that if

$$b \times (-a) = -ba \text{ as already shown,}$$

then

$$(-a) \times b = -ba$$

Likewise

$$a \times (-b) = -ba$$

and

$$(-b) \times a = -ba$$

These results lead to the following rule:

To multiply a quantity by a negative number, multiply it by the absolute value of the multiplier and change the sign of the result.

Now let us consider the operation of multiplying $(-a)$ by $(-b)$ in the light of this rule. Multiplying the quantity $(-a)$, by the absolute value of the multiplier, $(-b)$, we have $b \times (-a)$ or $-ba$, and changing the sign of this result, we get $+ba$. That is

$$(-b) \times (-a) = +ba, \text{ or simply } ba$$

Summarizing all of our results, we have—

$$(+b) \times (+a) = +ba$$

$$(+b) \times (-a) = -ba$$

$$(-b) \times (+a) = -ba$$

and $$(-b) \times (-a) = +ba$$

We may then set up the following general rule for signs in multiplication:

The product of two quantities having like signs is positive and the product of two quantities having unlike signs is negative.

4d. Division

Division in Algebra, as in Arithmetic, is the process of finding one of two numbers when their product and the other number are given. That is to say, if

$$a \times b = c$$

then $$c \div b = a$$

and $$c \div a = b$$

The result of the process is called the **quotient.** Division, then, is the inverse of multiplication and the rule of signs is very similar as may be seen from the following:

We know that $(+5) \times (+2) = (+10)$

and, since division is the inverse process to multiplication, then

$(+10) \div (+2) = (+5).$

Likewise $(-5) \times (+2) = (-10)$

and therefore $(-10) \div (+2) = (-5);$

$(+5) \times (-2) = (-10)$

and $(-10) \div (-2) = (+5);$

$(-5) \times (-2) = (+10)$

and $(+10) \div (-2) = (-5).$

We may accordingly write the rule—

The quotient of two quantities having like signs is positive and the quotient of two quantities having unlike signs is negative.

5. Examples

1. Add the following:

   (a) $+8$ and $+3$

   (b) $4a$ and $7a$

   (c) -14 and 18

   (d) $6a$ and $-3a$

   (e) $-9a$ and $2a$

   (f) $-a$ and a

   (g) $-a$ and $-4a$

   (h) $2a$ and $3b$

2. Subtract the following:

   (a) 11 from 19

   (b) -11 from 19

   (c) $6a$ from $8a$

   (d) $2a$ from $-11a$

   (e) $-14a$ from $8a$

   (f) a from $-a$

   (g) $-7a$ from $-6a$

   (h) 11 from $-2a$

3. Multiply the following:

   (a) 16 and -4

   (b) $2a$ and $11b$

   (c) $-a$ and 3

   (d) $6a$ and -10

   (e) $-9a$ and $-8b$

   (f) $2a$ and a

   (g) $6b$ and $-9b$

   (h) $3a$ and $-a^2$

4. Divide the following:

   (a) 6 by 2

   (b) -6 by 3

   (c) $12a$ by -4

   (d) $-18a$ by $-12a$

   (e) $10a^2$ by $2a$

   (f) a^2 by $-a$

   (g) $2a$ by $-a^2$

II. ALGEBRA—Addition, Subtraction, Multiplication, Division

6. Addition and Subtraction.

We have studied the processes involved in the fundamental operations of addition and subtraction when dealing with numbers or with a single symbol representing some number or quantity. We know that $6 + 4 + 2 = 12$ and that $6a + 4a + 2a = 12a$ where a represents, say, \$1.00 or any other definite thing or unit of measurement. Now consider the result if we attempt similarly to add different kinds of quantities as, for instance, 6 receivers + 4 automobiles + 2 testboards. We may say that there are 12 "things" or "objects" or something equally indefinite, but the sum is certainly not 12 receivers nor 12 automobiles nor 12 testboards. It is quite evident that we cannot add together unlike quantities and obtain a result that has any meaning.

Suppose that we assign symbols for the different things, using R to designate receivers, A to represent automobiles, and T for testboards. Our problem may then be algebraically expressed as follows:

$$6R + 4A + 2T = ?$$

We are unable to effect any solution because of the dissimilarity of the things to be added and we can only say that

$$6R + 4A + 2T = 6R + 4A + 2T.$$

If, however, we find two additional receivers, 2R, we may add them together with the six receivers, 6R, because these are like quantities. Thus—

$$6R + 4A + 2T + 2R = 8R + 4A + 2T$$

To illustrate further, assume that an equipment attendant taking an inventory of jacks in a small office finds the following:

1960 — # 218 jacks		
30 — # 216 "	}	in the Testroom
120 — # 49 "		
2405 — # 49 "	}	in the Operating Room
300 — #218 "		
25 — #218 "		
2 — #216 "	}	in Supplies
49 — # 49 "		

If we represent #218 jacks by a, #216 jacks by b, and #49 jacks by c, we may indicate the result of the inventory as follows:

$$\begin{array}{rrrl} 1960a & + 30b & + 120c & \text{(in Testroom)} \\ 300a & & + 2405c & \text{(in Operating Room)} \\ 25a & + 2b & + 49c & \text{(in Supplies)} \\ \hline \end{array}$$

And adding, we get $2285a + 32b + 2574c$ as the total number of jacks of each kind in the office. Note that we have added together only like quantities; there remain in the result three distinct kinds of quantities, which we cannot combine further.

In expressions such as the sum just found, the quantities separated by $+$, or any other operation or relation signs, are called **terms.** $2285a$, $32b$ and

$2574c$ are **unlike terms** and **cannot be added or subtracted**; $300a$, $25a$ and $1960a$ are **like terms** and **may be either added or subtracted.**

In the term, $2574c$, 2574 is called the **coefficient** of c and indicates the number of times that the letter is taken. This is a **numerical coefficient.** Similarly, in the purely letter term, abc, if we are interested in c, we say the coefficient of c is ab; in the same way, the coefficient of a is bc, the coefficient of b is ac and the coefficient of the combination, abc, is one.

In adding or subtracting algebraic quantities, the coefficient of any term in the sum or difference is equal to the sum or difference of the coefficients of the respective like terms in the original expression. Consider, for example, the expression

$$9a + 2b + 4ab + 3b - a^2 - 4a + 6ab = ?$$

Like terms are $9a$ and $-4a$, $2b$ and $3b$, $4ab$ and $6ab$; these and only these can be combined. The expression, then, is equal to

$$5a + 5b + 10ab - a^2$$

In the same way,

$$2x + 4xy - 8x^2 - 2x^2 + y - 8xy + x =$$
$$3x + y - 10x^2 - 4xy;$$
$$-a + b + c - a - b + c = -2a + 2c;$$

and so on.

7. Removal of Parentheses

In expressions containing several terms, it is a frequent practice to group the terms together in various ways by the use of parentheses or other aggregation signs. Before simplifying such ex-

pressions, it is usually necessary to remove the parentheses. Consider, for example, the expression, $10 + (5 + 1)$. This means that 5 and 1 are to be added together and the sum added to 10. The result is 16 and this is identical with the result of adding 5 and 1 separately to 10. That is—

$$10 + (5 + 1) = 16 = 10 + 5 + 1$$

We can remove the parenthesis in this case, therefore, without changing the value of the expression.

Now let us examine the expression, $10 - (5 + 1)$, which means that the sum of 5 and 1 is to be subtracted from 10. The result is 4 which is what we will obtain it we subtract 5 and 1 separately from 10. In other words—

$$10 - (5 + 1) = 4 = 10 - 5 - 1.$$

Again, $10 - (5 - 1) = 10 - 4 = 6 = 10 - 5 + 1.$

In both these examples, we see that in order to remove the parenthesis and have the value of the expression unchanged, it is necessary to change the sign of each of the terms within the parenthesis.

We may accordingly write the following rules for removing parentheses:

1. **If the parenthesis is preceded by a + sign, it may be removed without changing the signs of the enclosed terms.**
2. **If the parenthesis is preceded by a — sign, it may be removed only if the sign of each of the enclosed terms is changed.**

These rules are, of course, equally applicable to any of the other signs of aggregation.

When groups of terms are enclosed within other groups, all of the aggregation signs may be removed by the above rules. To do this, the innermost parenthesis is best removed first; the unravelling should be from the inside to the outside. As an illustration, let us simplify the following expression:

$$4a-\{2a+b+[6a-16-(-2-3b)+8b]-2\}=$$

(removing first the inner parenthesis)

$$4a-\{2a+b+[6a-16+2+3b+8b]-2\}=$$

(next removing the brackets)

$$4a-\{2a+b+6a-16+2+3b+8b-2\}=$$

(then the braces)

$$4a-2a-b-6a+16-2-3b-8b+2=$$

(and finally, making the indicated additions and subtractions)

$$-4a - 12b + 16.$$

8. Multiplication

We learned in Article 4c that $a \times b = ab$, $-a \times b = -ab$, $-a \times -b = ab$, etc. From this it follows that

$$a \times b \times c = ab \times c = abc$$

or $-a \times b \times -c = -ab \times -c = abc$, and so on.

The correct sign for the ultimate product may be obtained by multiplying two terms of a service together, giving this product the correct sign and then multiplying it by the next term and so on un-

til all the terms appear in the final product. Or, if more convenient, it can be remembered that **when the number of negative terms is even, the product is positive, and when the number of negative terms is odd, the product is negative.**

If we wish to multiply a group of terms, connected by + or — signs, by a single term, we must multiply each term in the group by the single term and add the separate products. Thus

$$a\times(b+c)=a(b+c)=a\times b+a\times c=ab+ac$$

$$2(6+4)=2\times10=20=2\times6+2\times4=12+8$$

$$m\ (n-l-p)=mn-ml-mp.$$

$$a\ [b+c\ (d+e)]=a\ [b+cd+ce]=ab+acd+ace$$

$$a\ \{b-c\ [d+f\ (m-p)]\}=a\ \{b-c\ [d+fm-fp]\}=a\ \{b-cd-cfm+cfp\}=ab-acd-acfm+acfp.$$

Note that in the last two examples, the multiplication indicated by the innermost parenthesis was performed first, followed by an addition or subtraction, another multiplication and so on.

In the same way, to multiply an expression containing more than one term by another expression also containing more than one term, we must multiply each term of the multiplicand by each term of the multiplier and add the separate products. Thus

$$(4+5)\times(2-3)=4\ (2-3)+5\ (2-3)$$
$$=8-12+10-15=-9,$$

a result which we can confirm by first evaluating the multiplier and multiplicand and then multiplying, ie., $4+5 = 9$, $2-3 = -1$ and $9 \times -1 = -9$.

Similarly,

$$(a + b) \times (c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd.$$

Now consider this example—

$$(2a^2 + b)\ (a^2 - b).$$

Multiplying as indicated above, we have

$$2a^2\ (a^2 - b) + b\ (a^2 - b) = 2a^2a^2 - 2a^2b$$
$$+ a^2b - bb$$

But from the definition of an exponent in Article 2b, a^2 means simply aa. Rewriting the product accordingly and combining the second and third terms since they are like terms, we have

$$2aaaa - a^2b - bb$$

Again by our definition of an exponent, $aaaa = a^4$ and $bb = b^2$.

Therefore our final result is

$$2a^4 - a^2b - b^2$$

Generalizing from the fact that the first term, $2a^4$, is the product of $2a^2$ and a^2, we may say that **the exponent of a term in the product is equal to the sum of the exponents of the like terms in the multiplier and multiplicand.** Thus also $b \times b = bb = b^2$, the exponents of the b terms being understood as 1 and adding together to give 2, the exponent of the product.

It is sometimes convenient to set up an example in multiplication in a form somewhat similar to that used in Arithmetic. As an illustration, the product of $a^3 + a^2b + ab^2 + b^2$ and $a + b$ is found below by this method:

Multiplicand	$a^3 + a^2b + ab^2 + b^3$	(1)
Multiplier	$a + b$	(2)
	$a^4 + a^3b + a^2b^2 + ab^3$	$[a \times (1)]$
	$a^3b + a^2b^2 + ab^3 + b^4$	$[b \times (1)]$
Product	$a^4 + 2a^3b + 2a^2b^2 + 2ab^3 + b^4$	[adding like terms]

9. Division

Division, we have learned, is the inverse process to multiplication. For instance, to divide ab by b we must find a quotient which when multiplied by the divisor b will give the dividend, ab. This quotient is obviously a. The solution to such a problem is in general most apparent if we write the problem in the fractional form, ie., ab/b instead of $ab \div b$. In the fraction, it is evident that the identical letter, b, occurring in both numerator and denominator, can be cancelled leaving the quotient, a. Similarly—

$$xy \div xy = \frac{xy}{xy} = 1$$

$$4\,abc \div 2a = \frac{4abc}{2a} = 2bc$$

$$a^2 \div a = \frac{aa}{a} = a$$

$$a^6 \div a^3 = \frac{aaaaaa}{aaa} = aaa = a^3$$

In the last two examples, we again deal with exponents and by expanding the terms, in accordance with the definition, we find that $a^2 \div a = a$ and $a^6 \div a^3 = a^3$. Analyzing these results, we see that in division **the exponent of a quantity in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.**

To divide an expression containing several terms by a single term, we divide each term of the dividend by the divisor and add the separate quotients. Thus, for example—

$$(12a^2b^4 - 9ab^3 + 6a^3b^4) \div 3ab^3 =$$

$$\frac{12a^2b^4}{3ab^3} + \frac{-9ab^3}{3ab^3} + \frac{6a^3b^4}{3ab^3} =$$

$$4ab - 3 + 2a^2b.$$

When both dividend and divisor contain more than one term, we must use a process quite similar to long division in Arithmetic. After first arranging the terms in both divisor and dividend in the order of powers of some common letter with the term having the highest power first, the procedure may be summarized as follows:

1. Divide the first term of the dividend by the first term of the divisor; the result is the first term of the quotient.

2. Multiply the entire divisor by this term and subtract the result from the dividend.

3. Treat the remainder as a new dividend and repeat the first two steps.

4. Continue the process until there is no remainder.

An arithmetic example will perhaps best illustrate the process. Let us divide 2475 by 11 but instead of the usual method, we shall write the dividend and divisor in the forms $2000 + 400 + 70 + 5$ and $10 + 1$ respectively. Then, to divide, we have

$$\begin{array}{lll}
10+1 \;/\; 2000+400+70+5 & / & 200+20+5=225 \\
\underline{2000+200} & & [200\times(10+1)] \\
200+70+5 & & \text{[Subtracting]} \\
\underline{200+20} & & [20\times(10+1)] \\
50+5 & & \text{[Subtracting]} \\
\underline{50+5} & & [5\times(10+1)] \\
0 & &
\end{array}$$

Now in the same way, let us divide $2x^3 - x^2 + 3x - 9$ by $2x - 3$. We proceed as follows:

$$\begin{array}{lll}
2x-3 \;/\; 2x^3-x^2+3x-9 & / & x^2+x+3 \\
\underline{2x^3-3x^2} & & [x^2\times(2x-3)] \\
2x^2+3x-9 & & \text{[Subtracting]} \\
\underline{2x^2-3x} & & [x\times(2x-3)] \\
6x-9 & & \text{[Subtracting]} \\
\underline{6x-9} & & [3\times(2x-3)] \\
0 & &
\end{array}$$

Note that each term of the quotient is obtained by dividing the first term in the corresponding dividend by the first term in the divisor. That is, $+x^2 - 2x^3 \div 2x$, $+x - 2x^2 \div 2x$ and $+3 - 6x \div 2x$.

10. Examples

1. Combine the following:

   (a) $150 + 2a + b + 2b$

   (b) $150 + 200 + x - 3x - 125$

   (c) $a^2 + 2ab + b^2 + 3a^2 - ab - 4b^2$

   (d) $3\frac{x}{y} - 6\frac{x^2}{y^2} - 7\frac{x}{y} + 12\frac{x^2}{y^2} + 10\frac{x}{y}$

   (e) $3(a + b) + 6(a + b) + c^2$

   (f) $a^2 + 1 - a^2 + c + 6 + 5a^2$

   (g) $abc + 3abc + 4cba - 5bac - c^2 - a^2$

   (h) $3\frac{ac}{b} - 6\frac{a}{b} + 3\frac{ac}{b} + 16\frac{a}{b} + b^2$

2. Remove parentheses from and combine the following:

   (a) $14 + (6 - 3)$

   (b) $24 - (-3 + 6)$

   (c) $6a - (b + 3)$

   (d) $13xy + (2ab + c)$

   (e) $5a - 4b + 3c - (-3a + 2b - c)$

   (f) $5a - [6 + 3 + (a - b)]$

(g) $m - [4n - k - (m + n - 2k)]$

(h) $3x-[7a-(5a-b-a)]-(-a-4b)$

(i) $3x + \{2y - [5x - (3y + x - 4y)]\}$

(j) $7cx-\{4cy-[4cx+3cy-(cy-cx)]\}$

3. Multiply the following

(a) $2a + b$ and 5

(b) $15a - b$ and 6

(c) $-a + a^2$ and b

(d) $17y + x$ and a

(e) $a^3 + a^2$ and a^2

(f) $3ab^2m^3 + 4a^2b - 2$ and $a^6b^7m^8$

(g) $3x + y$ and $x + y$

(h) $x + a$ and $x + a$

(i) $x - a$ and $x - a$

(j) $x^2 + 2xy + y^2$ and $x + y$

(k) $x + 2$ and $x - 3$ and $x^2 + 4$

(l) $x\ (x^2 - y^2) - 2$ and $x + y$

(m) $6\ (a + b)$ and $5\ (a + b)$

4. Divide the following

(a) $12m^2n$ by $4n$

(b) $30x^6y^5bc^3$ by $-6x^5y^5c^2$

(c) $-44a^3b^3c^3$ by $-11ab^2c^3$

(d) $-100x^4y^3z^2$ by x^3y^4

(e) $64m^2n^3 - 32mn^2 + 8m^2n$ by $8mn$

(f) $27x^3y^2z - 9x^3yz^2 - 33x^3y^2z^2$ by $-3x^3yz$

(g) $10(x + y)^2 - 5a(x + y) + 5a^2(x + y)$ by $5(x + y)$

(h) $x^2 - 7x + 12$ by $x - 3$

(i) $x^4 - 6xy - 9x^2 - y^2$ by $x^2 + y + 3x$

(j) $x^6 - 1$ by $x - 1$

III. ALGEBRA—Simple Equations

11. Definitions

An equation is the statement of an equality in the language of Algebra. Thus we may state the known fact that two added to two equals four in the form

$$2 + 2 = 4$$

and this is an equation. Similarly the statement that the area of a triangle is equal to one half the length of the base multiplied by the height can be expressed mathematically in the form

$$A = \tfrac{1}{2}\, bh$$

where A represents the area, b the length of the base and h the height. Such an equation or formula has the advantage of brevity and conciseness and it may be readily applied to a particular problem by substituting for the letters the values of the known dimensions.

In any equation the expression to the left of the = sign is called the **first member** or **left side** of the equation and the expression to the right is the **second member** or **right side.** Ordinarily an equation expresses the relation of one or more unknown quantities to certain known quantities and by means of this known relationship, we can usually determine the value of the unknown quantities.

12. Properties of Equations

To find the value of an unknown quantity in an equation or, as it is more commonly expressed, **to solve the equation,** frequently requires that the

equation be rewritten in a different form without destroying the equality of the two sides. By analyzing the properties of the equation, we can determine what operations may be performed on it without affecting its integrity.

Consider the equation, $6 - 2 = 4$. If we add 2 to both sides of the equation, we get $6 - 2 + 2 = 4 + 2$ or, cancelling the $+ 2$ and $- 2$ in the left side, $6 = 4 + 2$. This is obviously still a true statement. In the same way, if $x - 2 = 4$,

then $$x - 2 + 2 = 4 + 2$$

or $$x = 4 + 2 = 6,$$

the truth of which may be verified by substituting 6 for x in the initial equation. We may say, then, that **we may add the same quantity to both sides of an equation without destroying the equality.**

Since subtraction is really the addition of negative quantities, it follows from the above that **the same quantity may be subtracted from both sides of an equation without disturbing the equality.** For example take the equation, $6 + 2 = 5 + 3$, and subtract 2 from both sides. We have then $6 + 2 - 2 = 5 + 3 - 2$ or $6 = 5 + 3 - 2$ which is evidently true. Similarly subtracting 2 from both sides of $x + 2 = 4$, we get

$$x + 2 - 2 = 4 - 2$$

or $$x = 4 - 2 = 2$$

which may be verified by substitution in $x + 2 = 4$

Next consider the effect of multiplying both sides of an equation by the same quantity. If, for instance, we multiply both sides of $3 + 6 = 9$ by 3, we have.

$$3\ (3 + 6) = 3 \times 9$$

or $9 + 18 = 27$ which is correct. Likewise

multiplying $x - 2 = 4$ by 2

gives $2\ (x - 2) = 8$

or $2x - 4 = 8$

which is evidently true if $x = 6$, the value that it must have to satisfy the initial equation. Now let us divide both sides of $2x - 4 = 8$ by 2. We have

$$\frac{2x - 4}{2} = \frac{8}{2}$$

or $x - 2 = 4$ which is our original equation.

We may safely say, then, that **both sides of an equation may be multiplied or divided by the same quantity without affecting the equality.**

It follows that if $x + 3 = 7$,

then $(x + 3)\ (x + 3) = 7\ (x + 3)$, multiplying both sides by $(x + 3)$. But since $x + 3 = 7$, we may substitute 7 in the right side of the equation in place of $x + 3$ and have

$$(x + 3)\ (x + 3) = 7 \times 7$$

or $(x + 3)^2 = 7^2$

Substituting for x its value in the initial expression, namely 4, we get $(4 + 3)^2 = 7^2$

or $49 = 49$ which verifies the truth of the equation. If we take the square root of both sides of the equation $(x + 3)^2 = 49$, we get, of course, the initial equation, $x + 3 = 7$. Similar-

ly, if $x^6 + x^2 = 108$, then $\sqrt[3]{x^6 + x^2} = \sqrt[3]{108}$. The principle therefore may be stated—**raising both sides of an equation to the same power, or extracting the same root of both sides of an equation, does not disturb the equality.**

Now let us consider the two following equations:

$$a + b = x \qquad (1)$$

and

$$c + d = y \qquad (2)$$

If we add y to both sides of the first, we have

$$a + b + y = x + y; \qquad (3)$$

but from (2), $y = c + d$, and substituting this value in the left side of (3), we have

$$a + b + c + d = x + y.$$

This is the identical result we would get by adding the first member of (1) to the first member of (2) and the right side of (1) to the right side of (2) and equating the two sums. This process is called adding the two equations. It is evident, then, **that the result of adding two equations together is a third true equation.**

By going through a process similar to the above, it could also be shown in the same way that **the subtraction, multiplication or division of two equations results in a third true equation.**

Summarizing, we may write that if

$$a + b = x$$

and

$$c + d = y$$

then (1) $a + b + c + d = x + y,$

(2) $(a + b) - (c + d) = x - y,$

$$(3) \quad (a + b)\ (c + d) = xy,$$

and $$(4) \quad \frac{a + b}{c + d} = \frac{x}{y}$$

13. Algebraic Transformations

From the principles outlined above, we can derive certain working rules for handling equations. Consider the equation, $3x - 5 = 12$. Let us add 5 to both sides of the equation; we get $3x - 5 + 5 = 12 + 5$. Striking out the -5 and $+5$ in the left side, we have $3x = 12 + 5$ and comparing this with the original equation, we find that the net result has been to transfer the -5 of the left side of the original equation to the right side of the final equation with its sign changed to $+$. Similarly, suppose we subtract 5 from both sides of the equation, $3x + 5 = 12$. We have $3x + 5 - 5 = 12 - 5$ or $3x = 12 - 5$. Here we have transferred $+5$ from the left side to the right side and changed its sign to $-$. We may say, therefore, that **any term may be transferred from one side of an equation to the other by changing its sign.**

Now consider the equation, $2x^2 - x + 3 = 4a - x$, in which the term, $-x$, appears in both sides. If we add $+x$ to both sides, we get

$$2x^2 - x + 3 + x = 4a - x + x.$$

But the $+x$ and $-x$ now appearing on both sides may be cancelled, leaving $2x^2 + 3 = 4a$. From this, we conclude that **identical terms (ie., terms having the same value and sign) appearing on both sides of an equation may be cancelled.**

Frequently an equation takes a fractional form as, for instance, the following:

$$x + \frac{x+4}{2} - \frac{3x-4}{5} = 10.$$

We can transform this to a more workable form by multiplying both sides by 10, giving

$$10\left(x + \frac{x+4}{2} - \frac{3x-4}{5}\right) = 10 \times 10$$

or $\quad 10x + 5(x+4) - 2(3x-4) = 100$

or $\quad 10x + 5x + 20 - 6x + 8 = 100$

This is called **clearing the equation of fractions** and to accomplish it, we **multiply both sides of the equation (or each term of the equation) by the product of all the denominators.**

It may be noted that in an equation in which each member consists of a single simple fraction such, for example, as

$$\frac{x}{4} = \frac{9}{2},$$

the process known as cross-multiplying, ie., multiplying the numerator of the left side by the denominator of the right side and the numerator of the right side by the denominator of the left side, gives the same result as the above rule. Thus if, following the rule, we multiply both sides of the equation by the product of the denominators, we have

$$8 \times \frac{x}{4} = 8 \times \frac{9}{2}$$

or $$2x = 36$$

which is identical with the result of cross-multiplying, ie.,

$$2 \times x = 4 \times 9$$

14. Solution of Simple Equations

Now let us apply the above rules and principles to the solution of a few simple equations, containing one unknown quantity. As a first example, take the equation of the preceding Article—

$$x + \frac{x + 4}{2} - \frac{3x - 4}{5} = 10$$

We found that, multiplying this by 10 in order to clear fractions, we got

$$10x + 5x + 20 - 6x + 8 = 100$$

Next, let us collect all of the terms containing x, the unknown, on the left side and all of the known terms on the right side, being careful to change the signs of any terms that we transfer from one side to the other.

This gives

$$10x + 5x - 6x = 100 - 20 - 8$$

Combining like terms, this becomes

$$9x = 72.$$

Now divide both sides by 9. The result is

$$\frac{9x}{9} = \frac{72}{9}$$

or $$x = 8$$

We can verify this result by substituting 8 for x in the original equation, thus—

$$8 + \frac{8 + 4}{2} - \frac{24 - 4}{5} = 10$$

$$8 + 6 - 4 = 10$$

$$10 = 10$$

The last being an identity, our solution is proven correct.

As a further example, let us solve the following for t:

$$\frac{18.5 - \frac{t}{20}}{\frac{2}{5}} = t + 35$$

First clearing the major fraction by multiplying both sides by 2/5, we get

$$18.5 - \frac{t}{20} = \frac{2}{5}t + 14$$

Then clearing the remaining fractions by multiplying by 5×20 or 100—

$$1850 - 5t = 40t + 1400$$

Next putting unknowns on the left and knowns on the right, we have

$$-5t - 40t = 1400 - 1850$$

Combining terms

$$-45t = -450$$

and finally, dividing by —45, we have as our final result

$$t = \frac{-450}{-45} = 10$$

As in the preceding example, the correctness of the result may be verified by substituting 10 for t in the original equation.

The steps taken in solving simple equations such as illustrated in the above two examples, may be summarized for convenience as follows:

1. **Clear the equation of fractions by multiplying each term by the product of all the denominators.**
2. **Collect all unknown terms in the left side and all known terms in the right side.**
3. **Combine like terms on both sides.**
4. **Divide both sides by the coefficient of the unknown quantity.**

14. Examples

Solve the following equations, verifying each solution:

1. $7x - 14 = 49$

2. $16 - 3x = 13 - 6x$

3. $3(4x - 5) + 6 = 1 + 2x$

4. $6(5 - 2x) = 6 - 2(x - 2)$

5. $\dfrac{2x}{3} - \dfrac{4x}{3} = 5 - \dfrac{3x}{4}$

6. $\dfrac{x+1}{3} - \dfrac{x+4}{5} = 16 - \dfrac{x+3}{4}$

7. $\dfrac{3x}{5} - \dfrac{x}{6} = \dfrac{26}{15}$

8. $(x - 3)(x + 5) = (x + 1)(2x - 3) - x^2$

9. $\dfrac{2(5x - 3)}{2(4x + 3)} = \dfrac{6}{5}$

10. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$

    Note: Consider R as the unknown and R_1, R_2 and R_3 as knowns.

11. $x^2 - 4x + 4 = 25$

12. $\sqrt{x - 1} = 4$

13. $\dfrac{2x+1}{6} - \dfrac{3x+6}{4x-8} - \dfrac{x-1}{3} = 0$

14. $\dfrac{2}{x+1} - \dfrac{1}{x-1} = \dfrac{1}{2x-2}$

16. Equations Containing More Than One Unknown Quantity

We have seen how an equation containing only one unknown quantity can be solved for the one value that will satisfy the equation when substituted in place of the unknown. If, on the other hand, we have a single equation containing two unknowns such, for example, as $4x - 2y = 12$, we will find that there are an unlimited number of sets of values that will satisfy the equation. If we assume, for instance, that $x = 1$, the equation becomes

$4 - 2y = 12$ from which $y = -4$;

if $x = 2$, we have $8 - 2y = 12$ or $y = -2$;

if $x = 3$, we have $12 - 2y = 12$ or $y = 0$;

and so on.

It is evident that each of these sets of values,

$x = 1,\ y = -4$

$x = 2,\ y = -2$

$x = 3,\ y = 0,$

and an unlimited number of additional sets are equally able to satisfy the equation. It follows that there is no definite solution possible and the equation is therefore said to be **indeterminate.**

But if we have two equations each containing the same two unknown quantities, we will find that there is one and only one set of values which will simultaneously satisfy both equations. Thus the

equations, $x + y = 5$ and $x - y = 3$, are both satisfied by the values, $x = 4$ and $y = 1$ and by no other set of values, although each equation considered separately can be satisfied by any number of sets of values. Such equations are called **simultaneous equations.**

In order to effect a solution of two or more simultaneous equations, however, it is necessary that the equations be **independent,** that is, that they express different relationships between the unknown quantities. $x + y = 5$ and $x - y = 3$ are independent simultaneous equations because they give two distinct relationships between x and y and there is one set of values that will at the same time satisfy both equations. On the other hand, it will be found that there are any number of sets of values which will simultaneously satisfy the two equations, $x + y = 5$ and $2x + 2y = 10$. The latter two equations are, therefore, not independent; as a matter of fact, they represent one relationship expressed in two different ways, since the second equation, $2x + 2y = 10$, is merely the first equation multiplied by 2.

17. Methods of Solving Simultaneous Equations

Simultaneous equations containing two or more unknown quantities are solved by combining them in such a way as to obtain an equation containing only one of the unknown quantities. Since the other unknown quantities may be thought of as having been eliminated from this final equation, the process is called **elimination.** In order to eliminate all unknown quantities except one, it is necessary to have as many independent equations as

there are unknown quantities. If there are two unknown quantities, we must have two independent equations; if there are three unknowns, there must be three equations, and so on.

There are three general methods of elimination used for solving simultaneous equations. Just which method can be employed to best advantage depends upon the form of the equations to be solved. The first step in the actual solution of a particular set of equations will therefore consist in studying the form of the equations and selecting the method best suited to their solution. Examples will serve best to illustrate the three methods of elimination. The first of these is called

17a. Elimination by Addition or Subtraction

Consider the independent simultaneous equations:

$$3x + 4y = 24 \qquad (1)$$

$$5x - 6y = 2 \qquad (2)$$

Let us attempt to obtain a third equation from which x shall be eliminated. If we multiply equation (1) by 5 and equation (2) by 3, we will have two equations in both of which the term $15x$ occurs and if we then subtract one from the other, the resultant equation will contain no x term. Thus—

$$5 \times (1) \qquad 15x + 20y = 120 \qquad (3)$$

$$3 \times (2) \qquad 15x - 18y = 6 \qquad (4)$$

$$(3)-(4) \qquad + 38y = 114 \qquad (5)$$

Now solving the simple equation in y, (5), we have

$$38y = 114$$

$$y = \frac{114}{38} = 3$$

We can now find the value of x by substituting 3 for y in either equation (1) or (2). Using (2), we have

$$5x - 6 \times 3 = 2$$

$$5x - 18 = 2$$

$$5x = 2 + 18 = 20$$

$$x = \frac{20}{5} = 4$$

Our solution, then, is $x = 4$, $y = 3$. The correctness of this may be verified by substituting these values for x and y in equations (1) and (2), thus—

$3 \times 4 + 4 \times 3 = 24$ (1)

or $12 + 12 = 24$

and $5 \times 4 - 6 \times 3 = 2$ (2)

or $20 - 18 = 2$

Our solution is proven to be correct. As an example of

17b. Elimination by Substitution

We may solve the following:

$5x - 2y = 1$ (1)

$4x + 5y = 47$ (2)

We can find a value of one unknown in terms of the other from either equation. Substituting this value

in place of the first unknown in the other equation will give an equation containing the second unknown only. Thus from equation (1) we can find a value for x in terms of y—

$$5x-2y = 1 \qquad (1)$$

$$5x = 1+2y$$

$$x = \frac{1+2y}{5} \qquad (3)$$

Now substituting this value for x in equation (2) we get

$$4\left(\frac{1+2y}{5}\right) + 5y = 47$$

which is an equation in y only and may therefore be solved to obtain the numerical value of y as follows:

$$\frac{4+8y}{5} + 5y = 47$$

$$4+8y+25y = 235$$

$$8y+25y = 235-4$$

$$33y = 231$$

$$y = \frac{231}{33} = 7$$

Then substituting this value in (3), we have

$$x = \frac{1+2y}{5} = \frac{1+14}{5} = \frac{15}{5} = 3$$

and our solution is $x = 3$, $y = 7$.

The third method of elimination is called

17c. Elimination by Comparison.

Let us solve

$$7x+2y = 20 \qquad (1)$$

$$13x-3y = 17 \qquad (2)$$

If we find a value for one unknown in terms of the other from each equation and equate these two values, we will have an equation containing only the second unknown. Thus from (1) we may find the value of y in terms of x—

$$7x+2y = 20 \qquad (1)$$

$$2y = 20-7x$$

$$y = \frac{20-7x}{2} \qquad (3)$$

Similarly, from (2)

$$13x-3y = 17 \qquad (2)$$

$$-3y = 17-13x$$

$$3y = 13x-17$$

$$y = \frac{13x-17}{3} \qquad (4)$$

Now equating the two values for y given in (3) and (4), we have

$$\frac{20-7x}{2} = \frac{13x-17}{3}$$

and solving this for x—

$$3(20-7x) = 2(13x-17)$$

$$60-21x = 26x-34$$

$$-21x-26x = -34-60$$

$$-47x = -94$$

$$x = \frac{-94}{-47} = 2$$

Finally, substituting 2 for x in equation (3), we get

$$y = \frac{20-7x}{2} = \frac{20-14}{2} = \frac{6}{2} = 3$$

and our solution is $x = 2$, $y = 3$.

18. Solution of Equations Containing More than Two Unknowns

When the number of unknowns and the number of equations is greater than two, one of these same methods of elimination may be used to eliminate one unknown after another until an equation containing only one unknown is obtained. Consider, for example, the following set of equations containing three unknowns:

$$3x+4y-5z = 32 \qquad (1)$$

$$4x-5y+3z = 18 \qquad (2)$$

$$5x-3y-4z = 2 \qquad (3)$$

Let us first eliminate z between equations (1) and (2). To do this, multiply (1) by 3 and (2) by 5 and add, thus—

$3 \times (1)$ $\quad 9x+12y-15z = 96 \quad (4)$

$5 \times (2)$ $\quad 20x-25y+15z = 90 \quad (5)$

$(4)+(5)$ $\quad 29x-13y = 186 \quad (6)$

Now eliminate z between (2) and (3) by multiplying (2) by 4 and (3) by 3 and adding—

$4 \times (2)$ $\quad 16x-20y+12z = 72 \quad (7)$

$3 \times (3)$ $\quad 15x-9y-12z = 6 \quad (8)$

$(7)+(8)$ $\quad 31x-29y = 78 \quad (9)$

We now have two independent equations, (6) and (9), which each contain only the two unknowns, x and y. We may solve these as demonstrated in the preceding Article, in the following manner:

$$29x-13y = 186 \quad (6)$$

$$31x-29y = 78 \quad (9)$$

From (6) $\quad 29x = 186+13y$

or $$x = \frac{186+13y}{29} \quad (10)$$

Substituting this value in (9),

$$31\left(\frac{186+13y}{29}\right) - 29y = 78$$

$$\frac{5766+403y}{29} - 29y = 78$$

$$5766+403y-841y = 2262$$

$$-438y = 2262-5766 = -3504$$

$$y = \frac{-3504}{-438} = 8$$

Putting 8 in place of y in (10)

$$x = \frac{186+104}{29} = \frac{290}{29} = 10$$

We have now determined the values of x and y, and z may be found by substituting these in any one of the three original equations. Thus, using (1), we have

$$30+32-5z = 32$$
$$-5z = -30$$
$$z = \frac{-30}{-5} = 6$$

Our solution, then, is $x = 10$, $y = 8$, $z = 6$. These three values will satisfy each of the three original equations, which proves the solution to be correct.

As a further example, let us solve the set of equations given below containing six unknowns. This is a type of problem that is encountered in solving complex electrical networks for the currents in the different branches.

$$10-5I_1 -5I_2 -5I_6 = 0 \qquad (1)$$
$$10-5I_1 -6I_3 -10I_5 = 0 \qquad (2)$$
$$10-5I_1 -6I_3 -4I_4 -5I_6 = 0 \qquad (3)$$
$$I_1 = I_2 + I_3 \qquad (4)$$
$$I_3 = I_4 + I_5 \qquad (5)$$
$$I_6 = I_2 + I_4 \qquad (6)$$

Our method of attack will be to make use of the relationships given in equations (4), (5) and (6) to reduce the number of unknowns in equations (1), (2) and (3) to three. This done, we can effect a solution. First substituting in (1), (2) and (3) the value for I_1 given by (4), we get

$$10 - 5I_2 - 5I_3 - 5I_2 - 5I_6 = 0$$

$$10 - 5I_2 - 5I_3 - 6I_3 - 10I_5 = 0$$

$$10 - 5I_2 - 5I_3 - 6I_3 - 4I_4 - 5I_6 = 0$$

Combining like terms, these become

$$10 - 10I_2 - 5I_3 - 5I_6 = 0$$

$$10 - 5I_2 - 11I_3 - 10I_5 = 0$$

$$10 - 5I_2 - 11I_3 - 4I_4 - 5I_6 = 0$$

This is an improvement in that we now have only five unknowns, but the solution is still impossible so we will substitute for I_3 in the last three equations its value as given in equation (5).
We then have

$$10 - 10I_2 - 5I_4 - 5I_5 - 5I_6 = 0$$

$$10 - 5I_2 - 11I_4 - 11I_5 - 10I_5 = 0$$

$$10 - 5I_2 - 11I_4 - 11I_5 - 4I_4 - 5I_6 = 0$$

and these become, after combining like terms

$$10 - 10I_2 - 5I_4 - 5I_5 - 5I_6 = 0$$

$$10 - 5I_2 - 11I_4 - 21I_5 = 0$$

$$10 - 5I_2 - 15I_4 - 11I_5 - 5I_6 = 0$$

We still have four unknowns — one too many — so we will substitute the value of I_6 given in equation (6).

$$10-10I_2-5I_4-5I_5-5I_2-5I_4=0$$

$$10-5I_2-11I_4-21I_5=0$$

$$10-5I_2-15I_4-11I_5-5I_2-5I_4=0$$

and these, combining terms, reduce to

$$10-15I_2-10I_4-5I_5=0 \qquad (7)$$

$$10-5I_2-11I_4-21I_5=0 \qquad (8)$$

$$10-10I_2-20I_4-11I_5=0 \qquad (9)$$

Here we have three equations containing three unknowns, I_2, I_4 and I_5, and we can accordingly solve for their values. We can eliminate I_2 between (7) and (8) by multiplying the latter by 3 and subtracting—

$$10-15I_2-10I_4-5I_5=0 \qquad (7)$$

$3\times(8)\quad 30-15I_2-33I_4-63I_5=0$

Subtracting $-20+23I_4+58I_5=0 \qquad (10)$

Similarly eliminating I_2 between (8) and (9)

$2\times(8)\quad 20-10I_2-22I_4-42I_5=0$

$$10-10I_2-20I_4-11I_5=0 \qquad (9)$$

Subtracting $10-2I_4-31I_5=0 \qquad (11)$

Equations (10) and (11) each contain only the two unknowns, I_4 and I_5. We may rewrite these equations as follows:

$$23I_4+58I_5=20 \qquad (10)$$

$$-2I_4-31I_5=-10 \qquad (11)$$

Now, from (11) find the value of I_4 in terms of I_5. We have

$$-2I_4 = -10 + 31I_5$$

$$2I_4 = 10 - 31I_5$$

$$I_4 = \frac{10 - 31I_5}{2} = 5 - 15.5I_5 \qquad (12)$$

Substituting this value in (10) we get

$$23\ (5 - 15.5I_5) + 58I_5 = 20$$

$$115 - 356.5I_5 + 58I_5 = 20$$

$$-298.5I_5 = -95$$

$$I_5 = \frac{-95}{-298.5} = .3182$$

Next substituting this value for I_5 in (12) we get for I_4

$$I_4 = 5 - 15.5 \times .3182 = 5 - 4.9321 = .0679$$

Substituting the now known values of I_4 and I_5 in equation (7) we have

$$10 - 15I_2 - 10 \times .0679 - 5 \times .3182 = 0$$

$$10 - 15I_2 - .679 - 1.591 = 0$$

$$-15I_2 = -10 + .679 + 1.591 = -7.720$$

$$I_2 = \frac{-7.720}{-15} = .5147$$

The value of I_3 may be obtained from equation (5)

$$I_3 = I_4 + I_5$$

$$I_3 = .0679 + .3182 = .3861$$

From equation (6) we get

$$I_6 = I_2 + I_4 = .5147 + .0679 = .5826$$

and from equation (4)

$$I_1 = I_2 + I_3 = .5147 + .3861 = .9008$$

Our complete set of values, then, is

$$I_1 = .9008$$

$$I_2 = .5147$$

$$I_3 = .3861$$

$$I_4 = .0679$$

$$I_5 = .3182$$

$$I_6 = .5826$$

and these values, if correct, will satisfy all six of our original equations.

19. Examples

Solve by the addition or subtraction method:

1. $x+y = 17$
   $x-y = 7$
2. $x-12y = 3$
   $x+4y = 19$
3. $10x-3y = 25$
   $5x-9y = -25$
4. $5x+4y = 48$
   $2x+7y = 57$

Solve by the substitution method:

5. $x = 2y-3$

   $y = 2x-15$

6. $7x-3 = 5y$

   $7y-3 = 8x$

7. $3x+4y = 2$

   $9x+20y = 8$

8. $7x = 5y$

   $15y = 28x-70$

Solve by the comparison method:

9. $x = 3y-2$

   $x = 5y-12$

10. $5x+9y = 28$

    $7x+3y = 20$

11. $\frac{x}{7} + 7y = 99$

    $\frac{y}{7} + 7x = 51$

Solve the following:

12. $x + y = 28$

    $x + z = 30$

    $y + z = 32$

13. $4x-3y+2z = 9$

$2x+5y-3z = 4$

$5x+6y-2z = 18$

14. $10-10I_1 - 20I_2 = 0$

$10-10I_1 - 30I_3 = 0$

$I_1 = I_2 + I_3$

15. $10-10I_1 - 5I_2 - 5 = 0$

$10-10I_1 - 5I_2 - 5I_4 = 0$

$10-10I_1 - 10I_5 = 0$

$I_1 = I_2 + I_5$

$I_2 = I_3 + I_4$

V. LOGARITHMS

20. Properties of Exponents

The usefulness of logarithms as a convenient and time-saving tool in arithmetical calculations depends upon certain properties of exponents. We will therefore need to look into these properties before taking up the study of logarithms themselves.

We learned in Article 8 that in multiplying terms having exponents, we add the exponents of multiplier and multiplicand to get the exponent of the product. Thus

$$a^2 \times a^2 = a \times a \times a \times a = a^4 = a^{2+2}$$

Similarly,

$$10^2 \times 10 \times 10^3 = 100 \times 10 \times 1000 = 1000000 = 10^6$$
$$= 10^{2+1+3}$$

and $$a \times a^3 \times a^4 = a^{1+3+4} = a^8$$

Note that 1 is assumed as the exponent of the first term; any term when written without an exponent is considered as having the exponent 1, ie., $a = a^1$, $10 = 10^1$, etc.

In Article 9, we further found that in division we subtract exponents. For example,

$$10^8 \div 10^3 = 100{,}000{,}000 \div 1000 = 100{,}000$$
$$= 10^5 = 10^{8-3}$$

or $$\frac{a^6}{a^2} = \frac{aaaaaa}{aa} = aaaa = a^4 = a^{6-2}$$

It may be proven from this that any quantity having the exponent zero is equal to one.

Thus—

$$\frac{a^3}{a^3} = 1 = a^{3-3} = a^0$$

It can also be demonstrated that **a quantity having a negative exponent is equal to the reciprocal of the quantity with a positive exponent of the same value,** as follows:—

$$\frac{10}{1000} = \frac{1}{100} = \frac{1}{10^2} = \frac{10^1}{10^3} = 10^{1-3} = 10^{-2}$$

or, in general

$$\frac{a^2}{a^6} = a^{2-6} = a^{-4} = \frac{1}{a^4}$$

Now consider the result of raising an exponential term to some power, as for instance, $(10^3)^2$.

We have $(10^3)^2 = 1000^2 = 1000000 = 10^6 = 10^{3\times2}$

Similarly $(a^2)^4 = a^2 \times a^2 \times a^2 \times a^2 = a^8 = a^{2\times4}$

We may say, therefore, that **to raise an exponential term to a power, we multiply exponents.**

In like manner, let us find a root of an exponential term.

Taking $\sqrt{10^4}$ for an example,

we have $\sqrt{10^4} = \sqrt{10{,}000} = 100 = 10^2 = 10^{4\div2}$

Likewise $\sqrt[3]{a^6} = \sqrt[3]{a^2 \times a^2 \times a^2} = a^2 = a^{6\div3}$

It is evident, then, that **to take a root of an exponential term, we divide the exponent by the index of the root.**

21. Definition of Logarithm

A logarithm is simply an exponent. If we write $a^x = K$, we may say that x is the logarithm of K to the base a, or in other words, the power to which the base a must be raised to equal K.

In dealing with numbers, any positive, real number may be expressed as a power of 10 and it is convenient for a reason that will appear later to use this number, 10, as the base of **common** logarithms. Thus we know that $10^2 = 100$ and we may say, accordingly, that 2 is the logarithm of 100 to the base 10 ($\log_{10} 100 = 2$), or, if we adopt 10 as the standard base, simply that the logarithm of 100 is 2 ($\log 100 = 2$). Similarly the logarithm of 1000 is 3 and of 10 is 1 since $1000 = 10^3$ and $10 = 10^1$ while the logarithm of 1 is 0 because $10^0 = 1$. Obviously then, common logarithms of numbers between 1 and 10 have values between 0 and 1; logarithms of numbers between 1 and 10 have values between 1 and 2; numbers between 100 and 1000 have logarithms between 2 and 3, and so on. Logarithms of fractions or numbers less than one are negative. Thus the logarithm of .01 or $\frac{1}{100}$, for example, is -2 since $\frac{1}{100} = \frac{1}{10^2} = 10^{-2}$. In the same way, the logarithm of .1 is -1 since .1 =

$\frac{1}{10} = \frac{1}{10^1} = 10^{-1}$; and since $.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$, its logarithm is —3, and so on. Negative numbers do not have logarithms.

Other bases than 10 are sometimes used for systems of logarithms. The most important of these is that of the **natural** or Napierian system, 2.718+, commonly indicated by the Greek letter *e*. For certain kinds of mathematical work, the logarithmic system having this base has marked advantages, but for use in making arithmetical calculations it is decidedly inferior to the common system. Natural logarithms may be converted to common logarithms or vice versa by use of the following:

$$\log_{10} X = .434 \log_e X$$
$$\log_e X = 2.303 \log_{10} X$$

22. Logarithmic Tables

As has been noted above, the values of the common logarithms of all integral powers of the base 10 may be determined by observation, i.e., log 10 = 1, log 100 = 2, etc. The logarithms of the numbers lying between any of these integer powers of 10 may be determined by summing up certain infinite series, but we need not be concerned with the process because logarithmic tables are readily available from which we may obtain the common logarithms of all numbers to as great a degree of accuracy as we are likely to require. Common logarithms have the outstanding advantage over

logarithms to other bases, such as e, that the fractional or decimal part of the logarithm is the same for all numbers having the same sequence of figures. Thus the log of 1.432 is 0.15594, the log of 14.32 is 1.15594, log 143.2 equals 2.15594, and so on. This makes feasible the preparation of tables of common logarithms for **any** number whatever, whereas with a base such as e, where the logarithm must be determined separately for **every** number, only limited tables are practicable.

In the illustration above, we have log 14.32 = 1.15594. The integral part of the logarithm, i.e., 1., is called the **characteristic**; the fractional part, i.e., .15594, is called the **mantissa**. The characteristic is determined by observation and may be positive or negative, while the mantissa is taken from a table and is always positive. The determination of the value of the characteristic depends upon facts noted in the preceding Article. For instance, if we wish to find the logarithm of 186 we know that since this number is between 100 and 1000 (or between 10^2 and 10^3), its logarithm will be between 2 and 3, i.e., 2 plus a fraction. Its characteristic is therefore 2. Similarly, the characteristic of the logarithm of a fraction, as for instance, .0843, may be determined by considering that this number is between .1 and .01 (or, between 10^{-1} and 10^{-2}) and its logarithm is therefore between —1 and —2 and may be written as —2 plus a fraction, where —2 is the characteristic and the positive fraction is the mantissa.

The following simple rules may be used for determining the characteristic:

If the number is greater than 1, the characteristic is positive and is one unit less than the number of figures to the left of the decimal point.

If the number is less than 1, the characteristic is negative and is one unit more than the number of zeros between the decimal point and the first figure.

The logarithm of a number less than one may be written in several forms. We have noted, for example, that the logarithm of .0843 is —2 plus a fraction, and from the table we find the value of this fraction to be .92583. The logarithm may then be written as .92583—2 or as 2.92583 where the dash above the 2 means that it is negative; or the mantissa may be subtracted from 1 and the logarithm made entirely negative as —1.07417, but here it is to be noted that the fractional part is **not** that given in the table. For most purposes, it is customary to use the first form modified by adding to both terms a number sufficient to make the subtrahend 10 or an integer multiple of 10. Thus, by adding 8 to both terms of .92583—2 we have 8.92583—10 which is exactly the same in value and is somewhat more convenient in making calculations.

Tables of logarithms are available in a variety of sizes, the size of table required for any particular calculation depending entirely upon the degree of accuracy required. The table on pages 60, 61 and 62 is **four-place.** It gives accurately to four decimal places the mantissas of the logarithms of all numbers from 100 to 1000 and of all decimal fractions having less than four significant figures. Since the value of the mantissa depends only on the order of

figures in a number and is independent of the location of the decimal point, the logarithms of numbers consisting of only one or two significant figures may be readily obtained from this table. Thus the mantissa for the logarithm of 24 is the same as that for log 240 and the mantissa for log 8 is identical with that for log 800. The approximate value of logarithms of numbers having more than four significant figures may also be obtained from this table. For strict accuracy however, tables having five, seven or even more places may be required in such cases.

The method of using a logarithmic table to find the mantissa of the logarithm of a number will be apparent from an inspection of the table on pages 60 to 62. The mantissa is found at the intersection of the line designated at its left end with the first two figures of the number whose logarithm is being sought and the column headed by the third figure of the number. For example, let us find the logarithm of 8,760. From the table, we find the mantissa corresponding to 876 is .9425; the characteristic is positive and one less than the number of digits at the left of the decimal point, or 3. The logarithm is therefore, 3.9425.

Similarly, the characteristic for the log of 62 we know to be 1, and looking in the table under 620 we find the mantissa to be .7924; the logarithm is therefore 1.7924. Again, consider the fractional number .00178. From our rule we know that the characteristic is negative and one more than the number of zeros between the decimal point and the first figure, or —3. From the table, the mantissa corresponding to 178 is .2504. The log is then .2504 — 3 or, better, 7.2504 — 10.

When the logarithm of a number is known and it is desired to find the number, the process is, of course, the reverse of that described above. From the table the number corresponding to the mantissa of the known logarithm is first found and the position that the decimal point should occupy in this number is determined from the value of the characteristic. The rules for this follow from those given on page 58, thus—

> If the characteristic is positive, the number of figures to the left of the decimal point in the number is one more than the number of units in the characteristic.
>
> If the characteristic is negative, the number of zeros between the decimal point and the first figure of the fractional number is one less than the number of units in the characteristic.

Common Logarithms of Numbers—Four-place

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962

Common Logarithms of Numbers—Four-place

No.	0	1	2	3	4	5	6	7	8	9
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122

No.	0	1	2	3	4	5	6	7	8	9
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

To illustrate, let us find $\log^{-1} 1.7497$ (the number whose logarithm is 1.7497). Looking first in the table for a mantissa whose value is .7497, we find the corresponding number to be 562. The number

of digits to the left of the decimal point will be one greater than the characteristic, or two; the number sought is therefore, 56.2.

Similarly to find $\log^{-1} 7.6758\text{-}10$, we find the number corresponding to the mantissa, .6758 to be 474; the characteristic is 7—10 or —3 and there must be one less or two zeros to the right of the decimal point. The number, then, is .00474.

23. Uses of Logarithms

Since logarithms are really exponents, their usefulness will be evident from a study of the properties of exponents as given in Article 20. We saw there that in multiplication, we add exponents. It follows that **the logarithm of the product of two numbers is equal to the sum of the logarithms of the individual numbers.** That is:

$$\log (M \times N) = \log M + \log N$$

As an example, let us find the product, X, of 864 and .134. We have

$$\log X = \log 846 + \log .134$$

$$\log 846 = 2.9274$$

$$\log .134 = 9.1271 - 10$$

$$\log X = 12.0545\text{—}10 = 2.0545$$

$$X = \log^{-1} 2.0545 = 113. +$$

Note that the limitations of our table did not permit us to get an exact answer. By noting the fact, however, that the mantissa, .0545, lies approximately one third of the way between .0531, the mantissa for 113, and .0569, the mantissa for 114,

we can estimate the next figure in our product as 3 giving 113.3+. This last process is called **interpolation.**

We also noted in Article 20, that in division, we subtract exponents. From this it is quite evident that **the logarithm of the quotient of two numbers is equal to the difference of the logarithms of the individual numbers.**—

$$\log \frac{M}{N} = \log M - \log N$$

To illustrate this, we may find the quotient, X, of 64.85 divided by 264. From the rule just stated—

$$\log X = \log 64.85 - \log 264$$

$$\log 64.85 = 1.8119 = 11.8119 - 10$$

$$\log 264 = \quad 2.4216$$

$$\log X = \quad 9.3903 - 10$$

$$X = \log^{-1} 9.3903 - 10 = .2456$$

Here it was necessary to make use of interpolation both in finding the value of X from its logarithm and in finding the log of 64.85. In the latter case a value exactly intermediate between the mantissas for 648 and 649 was selected since the fourth figure was 5.

To raise an exponential term to a power we multiply exponents. Therefore, we may say that **the logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the number** or

$$\log M^n = n \log M$$

What is the value,X, of 583 raised to the 1.2 power i.e. $(583)^{1.2}$? From our rule, we have

$$\log X = 1.2 \log 583$$

$$\log 583 = 2.7657$$

$$\log X = 1.2 \times 2.7657 = 3.3188$$

$$X = \log^{1} 3.3188 = 2084.$$

Similarly, since we divide the exponent by the index when finding a root of an exponential term, it follows that **the logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.** In symbols—

$$\log \sqrt[n]{M} = \frac{\log M}{n}$$

Thus, to find, for example, the value, X, of $\sqrt[2.7]{.00373}$ we have

$$\log X = \frac{\log .00373}{2.7}$$

$$\log .00373 = 7.5717 - 10$$

$$\log X = \frac{7.5717-10}{2.7} = \frac{24.5717-27}{2.7}$$

$$= 9.1006-10$$

$$X = \log^{-1} 9.1006-10 = .1261$$

It is seen, then, that the fundamental purpose of logarithms is to reduce the amount of work involved in arithmetical calculations. Multiplication

and division are reduced to addition and subtraction respectively, and when the numbers to be multiplied or divided have a large number of significant figures, the time and effort consumed in finding their logarithms from a table, performing the necessary addition or subtraction and converting the resultant logarithm back to its corresponding number is less than would be required to go through the ordinary arithmetical process. The use of logarithms reduces involution and evolution to multiplication and division respectively and by virtue of this, permits certain processes that are impossible or exceedingly difficult of performance by Arithmetic.

If, for instance, we wished to find the value of a certain number raised to the 18th power, we could accomplish it in Arithmetic by multiplying the number together eighteen times; but the use of logarithms would greatly shorten the operation. If, however, the power to which we desired to raise the number were 18.43, the use of logarithms would be imperative, and the process would be no more difficult than the preceding one. The same considerations apply to processes involving the extraction of roots.

24. Examples

1. Evaluate the following:

   (a) $(x^3)^{10}$; (b) $\sqrt[4]{x^{20}}$; (c) x^{18}/x^{12}; (d) $\sqrt{(x^2)^4}$

2. Find the logs of the following:

   (a) 6000; (b) .0132; (c) 21.6; (d) .000031; (e) 1.03

3. **Determine the numbers whose logarithms are the following:**

   (a) 3.8791; (b) .0128; (c) 8.5192—10; (d) 3.000

4. Perform the following calculations using logarithms:

(a) $2{,}325{,}000 \times .453$

(b) $.143 \div 6.484$

(c) $(3.265)^5$

(d) $(.0218)^{2.6}$

(e) $\sqrt[3]{.00791}$

(f) $\sqrt[1.8]{467{,}500}$

5. Find the value of X in the following equation:

$$X = \frac{(7.142)^3 \times (.0174)^{.8}}{\sqrt[2.2]{24.65}}$$

VI. ELEMENTS OF TRIGONOMETRY

25. Review of Plane Geometry

Trigonometry has to do chiefly with angles and the relationships between angles and the sides of right triangles constructed on them. As a first step in its study, therefore, it may be well to review briefly those concepts of plane geometry which treat of these subjects.

25a. The Circle.

A **circle** is defined as a line so drawn about a point called the **center** that every portion of the line is the same distance from the point. Referring to Figure 1, the curved line is called the **circumference**; the straight line from the center to any point of the circumference is called the **radius**; and a straight

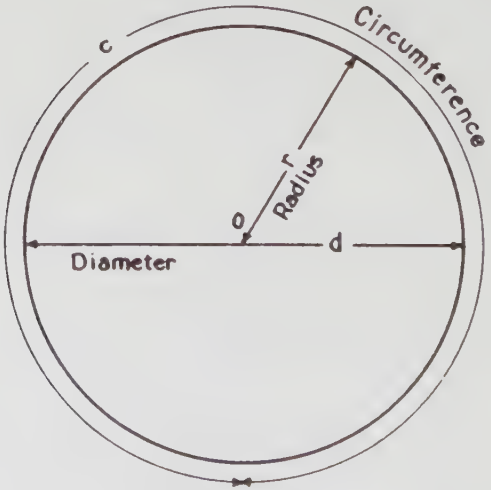


Figure 1

line drawn through the center and terminating at both ends in the circumference is called the **diameter.** Obviously, then, the diameter is equal to twice the radius, or $d = 2r$, where d represents the diameter and r the radius. The relation that the diameter bears to the circumference is given by the expression: $c/d = 3.14159$—where c represents the circumference. This ratio is usually indicated for convenience by the Greek letter π, i.e., $\pi = 3.1416$— From this relation and our definitions, we have—

$$c = \pi d = 2\pi r$$

The area of the circle may be shown to be—

$$A = \pi r^2 = \frac{\pi d^2}{4}$$

25b. Angular Measure

If a line starting from the position OX of Figure 2 is rotated about the point O to some other position such as that represented by OA, it is said to generate the angle XOA or θ. If the generating line is rotated in a **counter-clockwise** direction (i.e., opposite to the direction of motion of the hands of a clock), as in the figure, the angle is said to be **positive**; if the rotation had been in the opposite or **clockwise** direction, the angle would have been negative. The angle that will be generated by rotating OA through one complete revolution so that it is returned to its starting point is called **360 degrees** (360°). That is to say, the complete circular angle is divided into 360 equal small angles for convenience in making measurements. For greater

precision of measurement, each degree is divided into 60 equal parts called **minutes** (60′), and each minute is in turn divided into 60 equal parts called **seconds** (60″). Thus—

$$1^\circ = 60' = 60 \times 60 \text{ or } 3600''$$

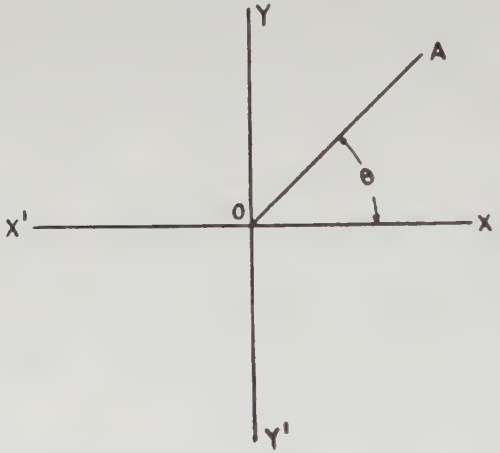


Figure 2

In Figure 2 it was convenient to consider the rotating line OA as starting from a horizontal axis X-X′. It is also desirable in such a diagram to draw through the point O about which the line rotates, a vertical axis Y-Y′. These two axes, being perpendicular to each other obviously divide the complete angle of revolution, 360°, into four equal angles each having a value of 360/4 or 90°. These 90° angles, XOY, YOX′, X′OY′ and Y′OX are called **right angles.**

If now we construct a circle with the point O as its center and having a radius equal to OA as in Figure 3, we may consider that the circumference

of the circle is divided into 360 equal parts by the radius OA as it rotates through 360 degrees of angle. We know that the circumference is equal to 2π times the radius or that the radius is "$1/2\pi$th" of the circumference. Then, if we consider the length of the circumference as being 360 units or degrees, the length of the radius will be $360/2\pi$ or

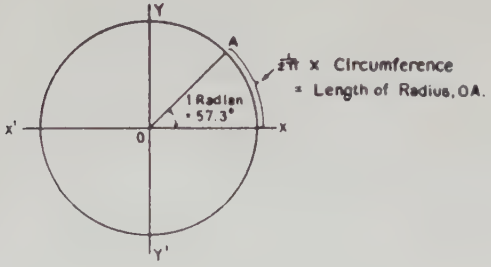


Figure 3

57.296+ of the same units. In certain mathematical work it is convenient to measure angles in this unit (57.3°), which is called a **radian** and 2π of which equal 360° or one complete angle of revolution.

25c. The Right Triangle

A triangle one of whose three interior angles is a right angle is called a **right triangle.** Such a triangle has certain properties which make it very useful in any work involving the measurement of distance, either actual or mathematical. Figure 4 illustrates a right triangle where side OX, or B, is known as the **base,** side XY, or A, is called the **altitude** and side OY, or C, is the **hypotenuse.** The

properties of this triangle that are especially useful may be stated as follows:

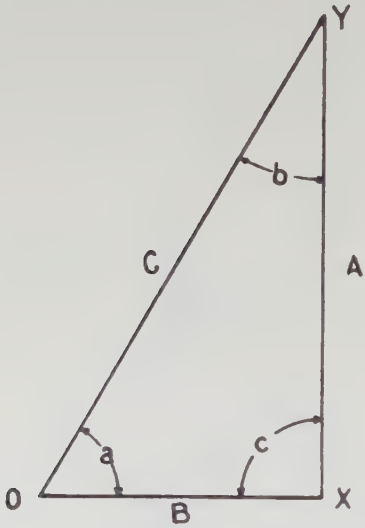


Figure 4

(a) Its area is equal to one-half the product of the legs—

$$\text{area} = AB/2$$

(b) The sum of the acute angles is equal to the right angle or 90°, ie., $\angle a + \angle b = \angle c = 90°$.

(c) As a corollary to (b), the acute angles are **complimentary,** $\angle a = 90° - \angle b$ and $\angle b = 90° - \angle a$.

(d) The square of the hypotenuse is equal to the sum of the squares of the two legs, ie., $C^2 = A^2 + B^2$.

The properties of this triangle form the basis of **trigonometry.**

26. Trigonometric Functions

If we construct two right triangles of different sizes but having either one of their acute angles equal, then all three angles of the two triangles will be equal and the triangles are said to be **similar.** Now we will find that while the areas and the lengths of the three sides of the two triangles may differ widely, the ratio between any two sides of one triangle will be identical in value with the ratio between the corresponding two sides of the other triangle and this value will depend on or **be a function of one of the acute angles.** Thus, consider the two triangles of Figure 5, one of which has legs 3′ and 4′ long and hypotenuse 5′ while the corresponding legs and hypotenuse of the other are 3″, 4″ and 5″. These two triangles are similar; angle X equals angle X′ and angle Y equals angle Y′.

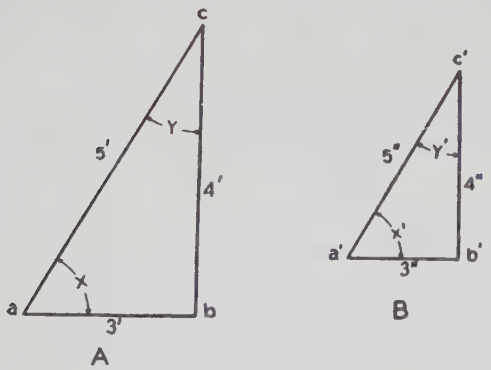


Figure 5

Now note that the ratio of side **ab** to side **ac** of triangle A is equal to 3/5 which equals the ratio of side **a'b'** to side **a'c'** of triangle B, or—

$$\frac{ab}{ac} = \frac{3}{5} = \frac{a'b'}{a'c'}$$

Similarly:

$$\frac{bc}{ab} = \frac{4}{3} = \frac{b'c'}{a'b'}$$

$$\frac{bc}{ac} = \frac{4}{5} = \frac{b'c'}{a'c'}$$

and so on, for all possible ratios. The value of each one of these ratios may be considered as being dependent on the value of the angle X or X', for if we change this angle, the relative length of the three sides will obviously be changed. This being so, we may prepare a single set of values for the ratios of the various sides which will apply to any right triangle whatever having one acute angle of value X. The values of these ratios are called the **trigonometric functions** of the angle X and have been given the following names:

$$\frac{ab}{ac} = \frac{\text{side adjacent to the angle X}}{\text{hypotenuse}} = \textbf{cosine X} \quad (1)$$

$$\frac{bc}{ab} = \frac{\text{side opposite the angle X}}{\text{side adjacent to the angle X}} = \textbf{tangent X} \quad (2)$$

$$\frac{bc}{ac} = \frac{\text{side opposite the angle X}}{\text{hypotenuse}} = \textbf{sine X} \quad (3)$$

The reciprocals of these three ratios, i.e., $\frac{ac}{ab}$, $\frac{ab}{bc}$ and $\frac{ac}{bc}$ are known as **secant X, cotangent X** and **cosecant X** respectively, but we need not ordinarily be much concerned with these latter functions.

These same ratios may also be considered as functions of the angle Y or Y′. Thus, applying the same definitions as in (1), (2) and (3) above, we have—

$$\cos Y = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{bc}{ac} = \sin X$$

$$\sin Y = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{ab}{ac} = \cos X$$

and

$$\tan Y = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{ab}{bc} = \cot X$$

Since the values of the ratios depend only on the size of the angle, a table may be prepared showing the values of these ratios or functions for all values of angle between 0° and 90° (The treatment of angles larger than 90° will be explained in the next Article). The table on pages 79 to 86 gives to an accuracy of four decimal places the values of the sines, cosines, tangents and cotangents of angles from 0° to 90° at 10′ intervals. Angles between 0° and 45° are listed in the column farthest to the left reading down and the values of their functions are given in the columns headed at the top of the page with the name of the function; angles between 45° and 90° are listed in the column farthest to the right reading up, and the same columns of functions

are used for these but the particular function to be found in each column is that indicated by the names at the foot of the columns.

Such a table of trigonometric functions is an extremely convenient tool for use in dealing with angular measurements. With its aid we may determine, for instance, the length of the other two sides of a right triangle when we know the length of the third side and the value of one of the acute angles; or, knowing the lengths of the sides, we can find the angles, and so on. Thus, referring to the triangle of Figure 4, let us assume that we know the values of sides A and B and wish to find the angles a and b and the value of C. If A = 7 and B = 4, we may say, since the tangent of an angle equals the side opposite divided by the side adjacent—

$$\tan a = \frac{7}{4} = 1.750$$

from which a equals the angle whose tangent is 1.750, or

$$a = \tan^{-1} 1.750 = 60^\circ\ 15' \text{ approximately.}$$

The angle b is, of course, equal to 90° minus this value,

$$b = 90^\circ - 60^\circ 15' = 29^\circ 45'$$

We could also get this result by considering that the tangent of b is equal to B divided by A, i.e.,

$$\tan b = \frac{4}{7} = .5714$$

and from the table—

$$b = \tan^{-1} .5714 = 29°45' \text{ approximately.}$$

To get C, we could use the fact that $C^2 = A^2 + B^2$ from which

$$C = \sqrt{A^2 + B^2} = \sqrt{16 + 49} = \sqrt{65} = 8.06 \text{ app.}$$

Or, knowing the value of a, we can instead make use of the fact that the cosine of a is equal to B, the side adjacent, divided by the hypotenuse, C, or

$$\cos a = \frac{B}{C} \text{ from which } C = \frac{B}{\cos a}$$

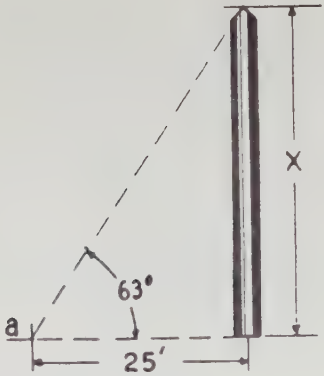
From the table, we find the value of cos a or cos 60°15′ to be about .4962. We then have

$$C = \frac{B}{\cos a} = \frac{4}{.4962} = 8.06 \text{ approximately}$$

With practice, it will be found that the use of a table of trigonometric functions to find the hypotenuse of a right triangle, given the two sides, is frequently more efficient and almost always more accurate than the conventional method of taking the square root of the sum of the squares.

Practical applications for the use of trigonometric functions are very common in all kinds of measuring work and in mathematical calculations involving "vector" quantities. An exceedingly practical application is that of finding the height of an object, such for example as a telephone pole, when a direct measurement is impracticable. To do this we measure any convenient horizontal distance from the base of the pole, say 25 feet. Next,

sight from the end of this line, a, to the top of the pole and measure with a protractor of some sort the angle between the horizontal and the line of vision. Suppose this is found to be 63°. Then,



knowing that the tangent of the angle is the side opposite divided by the side adjacent, we have

$$\tan 63^\circ = \frac{X}{25}$$

and referring to the table for the value of tan 63° we get

$$1.9626 = \frac{X}{25}$$

from which $X = 25 \times 1.9626 = 49$ ft. approximately.

ANGLE ° ′	SIN	COS	TAN	COT		
0 00	.0000	1.0000	.0000	Infinite	00	90
10	.0029	1.0000	.0029	343.77	50	
20	.0058	1.0000	.0058	171.89	40	
30	.0087	1.0000	.0087	114.59	30	
40	.0116	.9999	.0116	85.940	20	
50	.0145	.9999	.0145	68.750	10	
1 00	.0175	.9998	.0175	57.290	00	89
10	.0204	.9998	.0204	49.104	50	
20	.0233	.9997	.0233	42.964	40	
30	.0262	.9997	.0262	38.188	30	
40	.0291	.9996	.0291	34.368	20	
50	.0320	.9995	.0320	31.242	10	
2 00	.0349	.9994	.0349	28.636	00	88
10	.0378	.9993	.0378	26.432	50	
20	.0407	.9992	.0407	24.542	40	
30	.0436	.9990	.0437	22.904	30	
40	.0465	.9989	.0466	21.470	20	
50	.0494	.9988	.0495	20.206	10	
3 00	.0523	.9986	.0524	19.081	00	87
10	.0552	.9985	.0553	18.075	50	
20	.0581	.9983	.0582	17.169	40	
30	.0610	.9981	.0612	16.350	30	
40	.0640	.9980	.0641	15.605	20	
50	.0669	.9978	.0670	14.924	10	
4 00	.0698	.9976	.0699	14.301	00	86
10	.0727	.9974	.0729	13.727	50	
20	.0756	.9971	.0758	13.197	40	
30	.0785	.9969	.0787	12.706	30	
40	.0814	.9967	.0816	12.251	20	
50	.0843	.9964	.0846	11.826	10	
5 00	.0872	.9962	.0875	11.430	00	85
10	.0901	.9959	.0904	11.059	50	
20	.0929	.9957	.0934	10.712	40	
30	.0958	.9954	.0963	10.385	30	
40	.0987	.9951	.0992	10.078	20	
50	.1016	.9948	.1022	9.7882	10	
	COS	SIN	COT	TAN	′ ° ANGLE	

ANGLE °	′	SIN	COS	TAN	COT		
6	00	.1045	.9945	.1051	9.5144	00	84
	10	.1074	.9942	.1080	9.2553	50	
	20	.1103	.9939	.1110	9.0098	40	
	30	.1132	.9936	.1139	8.7769	30	
	40	.1161	.9932	.1169	8.5555	20	
	50	.1190	.9929	.1198	8.3450	10	
7	00	.1219	.9925	.1228	8.1443	00	83
	10	.1248	.9922	.1257	7.9530	50	
	20	.1276	.9918	.1287	7.7704	40	
	30	.1305	.9914	.1317	7.5958	30	
	40	.1334	.9911	.1346	7.4287	20	
	50	.1363	.9907	.1376	7.2687	10	
8	00	.1392	.9903	.1405	7.1154	00	82
	10	.1421	.9899	.1435	6.9682	50	
	20	.1449	.9894	.1465	6.8269	40	
	30	.1478	.9890	.1495	6.6912	30	
	40	.1507	.9886	.1524	6.5606	20	
	50	.1536	.9881	.1554	6.4348	10	
9	00	.1564	.9877	.1584	6.3138	00	81
	10	.1593	.9872	.1614	6.1970	50	
	20	.1622	.9868	.1644	6.0844	40	
	30	.1650	.9863	.1673	5.9758	30	
	40	.1679	.9858	.1703	5.8708	20	
	50	.1708	.9853	.1733	5.7694	10	
10	00	.1736	.9848	.1763	5.6713	00	80
	10	.1765	.9843	.1793	5.5764	50	
	20	.1794	.9838	.1823	5.4845	40	
	30	.1822	.9833	.1853	5.3955	30	
	40	.1851	.9827	.1883	5.3093	20	
	50	.1880	.9822	.1914	5.2257	10	
11	00	.1908	.9816	.1944	5.1446	00	79
	10	.1937	.9811	.1974	5.0658	50	
	20	.1965	.9805	.2004	4.9894	40	
	30	.1994	.9799	.2035	4.9152	30	
	40	.2022	.9793	.2065	4.8430	20	
	50	.2051	.9787	.2095	4.7729	10	
		COS	SIN	COT	TAN	′ ° ANGLE	

ANGLE °	′	SIN	COS	TAN	COT		
12	00	.2079	.9781	.2126	4.7046	00	78
	10	.2108	.9775	.2156	4.6382	50	
	20	.2136	.9769	.2186	4.5736	40	
	30	.2164	.9763	.2217	4.5107	30	
	40	.2193	.9757	.2247	4.4494	20	
	50	.2221	.9750	.2278	4.3897	10	
13	00	.2250	.9744	.2309	4.3315	00	77
	10	.2278	.9737	.2339	4.2747	50	
	20	.2306	.9730	.2370	4.2193	40	
	30	.2334	.9724	.2401	4.1653	30	
	40	.2363	.9717	.2432	4.1126	20	
	50	.2391	.9710	.2462	4.0611	10	
14	00	.2419	.9703	.2493	4.0108	00	76
	10	.2447	.9696	.2524	3.9617	50	
	20	.2476	.9689	.2555	3.9136	40	
	30	.2504	.9681	.2586	3.8667	30	
	40	.2532	.9674	.2617	3.8208	20	
	50	.2560	.9667	.2648	3.7760	10	
15	00	.2588	.9659	.2679	3.7321	00	75
	10	.2616	.9652	.2711	3.6891	50	
	20	.2644	.9644	.2742	3.6470	40	
	30	.2672	.9636	.2773	3.6059	30	
	40	.2700	.9628	.2805	3.5656	20	
	50	.2728	.9621	.2836	3.5261	10	
16	00	.2756	.9613	.2867	3.4874	00	74
	10	.2784	.9605	.2899	3.4495	50	
	20	.2812	.9596	.2931	3.4124	40	
	30	.2840	.9588	.2962	3.3759	30	
	40	.2868	.9580	.2994	3.3402	20	
	50	.2896	.9572	.3026	3.3052	10	
17	00	.2924	.9563	.3057	3.2709	00	73
	10	.2952	.9555	.3089	3.2371	50	
	20	.2979	.9546	.3121	3.2041	40	
	30	.3007	.9537	.3153	3.1716	30	
	40	.3035	.9528	.3185	3.1397	20	
	50	.3062	.9520	.3217	3.1084	10	
		COS	SIN	COT	TAN	′ ANGLE	°

ANGLE °		SIN	COS	TAN	COT		
18	00	.3090	.9511	.3249	3 0777	00	72
	10	.3118	.9502	.3281	3 0475	50	
	20	.3145	.9492	.3314	3.0178	40	
	30	.3173	.9483	3346	2.9887	30	
	40	.3201	.9474	.3378	2.9600	20	
	50	.3228	.9465	.3411	2.9319	10	
19	00	.3256	.9455	.3443	2.9042	00	71
	10	.3283	9446	.3476	2.8770	50	
	20	.3311	.9436	.3508	2 8502	40	
	30	.3338	.9426	.3541	2 8239	30	
	40	.3365	9417	.3574	2.7980	20	
	50	.3393	.9407	.3607	2.7725	10	
20	00	.3420	.9397	.3640	2.7475	00	70
	10	.3448	.9387	.3673	2.7228	50	
	20	3475	.9377	.3706	2.6985	40	
	30	3502	.9367	.3739	2.6746	30	
	40	.3529	.9356	.3772	2.6511	20	
	50	.3557	.9346	.3805	2.6279	10	
21	00	.3584	.9336	.3839	2.6051	00	69
	10	.3611	.9325	.3872	2.5826	50	
	20	.3638	.9315	.3906	2.5605	40	
	30	.3665	.9304	.3939	2.5386	30	
	40	.3692	.9293	.3973	2.5172	20	
	50	.3719	.9283	.4006	2.4960	10	
22	00	.3746	.9272	.4040	2.4751	00	68
	10	.3773	.9261	.4074	2.4545	50	
	20	.3800	.9250	.4108	2 4342	40	
	30	.3327	.9239	.4142	2.4142	30	
	40	.3854	.9228	.4176	2.3945	20	
	50	.3881	.9216	.4210	2.3750	10	
23	00	.3907	.9205	.4245	2.3559	00	67
	10	.3934	.9194	.4279	2.3369	50	
	20	.3961	.9182	.4314	2.3183	40	
	30	.3987	.9171	.4348	2.2998	30	
	40	.4014	.9159	.4383	2.2817	20	
	50	.4041	.9147	.4417	2.2637	10	
		COS	SIN	COT	TAN	′ ANGLE	°

ANGLE °		SIN	COS	TAN	COT		
24	00	.4067	.9135	.4452	2.2460	00	66
	10	.4094	.9124	.4487	2.2286	50	
	20	.4120	.9112	.4522	2.2113	40	
	30	.4147	.9100	.4557	2.1943	30	
	40	.4173	.9088	.4592	2.1775	20	
	50	.4200	.9075	.4628	2.1609	10	
25	00	.4226	.9063	.4663	2.1445	00	65
	10	.4253	.9051	.4699	2.1283	50	
	20	.4279	.9038	.4734	2.1123	40	
	30	.4305	.9026	.4770	2.0965	30	
	40	.4331	.9013	.4806	2.0809	20	
	50	.4358	.9001	.4841	2.0655	10	
26	00	.4384	.8988	.4877	2.0503	00	64
	10	.4410	.8975	.4913	2.0353	50	
	20	.4436	.8962	.4950	2.0204	40	
	30	.4462	.8949	.4986	2.0057	30	
	40	.4488	.8936	.5022	1.9912	20	
	50	.4514	.8923	.5059	1.9768	10	
27	00	.4540	.8910	.5095	1.9626	00	63
	10	.4566	.8897	.5132	1.9486	50	
	20	.4592	.8884	.5169	1.9347	40	
	30	.4617	.8870	.5206	1.9210	30	
	40	.4643	.8857	.5243	1.9074	20	
	50	.4669	.8843	.5280	1.8940	10	
28	00	.4695	.8829	.5317	1.8807	00	62
	10	.4720	.8816	.5354	1.8676	50	
	20	.4746	.8802	.5392	1.8546	40	
	30	.4772	.8788	.5430	1.8418	30	
	40	.4797	.8774	.5467	1.8291	20	
	50	.4823	.9760	.5505	1.8165	10	
29	00	.4848	.8746	.5543	1.8040	00	61
	10	.4874	.8732	.5581	1.7917	50	
	20	.4899	.8718	.5619	1.7796	40	
	30	.4924	.8704	.5658	1.7675	30	
	40	.4950	.8689	.5696	1.7556	20	
	50	.4975	.8675	.5735	1.7437	10	
		COS	SIN	COT	TAN	′ ANGLE	°

ANGLE °	′	SIN	COS	TAN	COT		
30	00	.5000	.8660	.5774	1.7321	00	60
	10	.5025	.8646	.5812	1.7205	50	
	20	.5050	.8631	.5851	1.7090	40	
	30	.5075	.8616	.5890	1.6977	30	
	40	.5100	.8601	.5930	1.6864	20	
	50	.5125	.8587	.5969	1.6753	10	
31	00	.5150	.8572	.6009	1.6643	00	59
	10	.5175	.8557	.6048	1.6534	50	
	20	.5200	.8542	.6088	1.6426	40	
	30	.5225	.8526	.6128	1.6319	30	
	40	.5250	.8511	.6168	1.6212	20	
	50	.5275	.8496	.6208	1.6107	10	
32	00	.5299	.8480	.6249	1.6003	00	58
	10	.5324	.8465	.6289	1.5900	50	
	20	.5348	.8450	.6330	1.5798	40	
	30	.5373	.8434	.6371	1.5697	30	
	40	.5398	.8418	.6412	1.5597	20	
	50	.5422	.8403	.6453	1.5497	10	
33	00	.5446	.8387	.6494	1.5399	00	57
	10	.5471	.8371	.6536	1.5301	50	
	20	.5495	.8355	.6577	1.5204	40	
	30	.5519	.8339	.6619	1.5108	30	
	40	.5544	.8323	.6661	1.5013	20	
	50	.5568	.8307	.6703	1.4919	10	
34	00	.5592	.8290	.6745	1.4826	00	56
	10	.5616	.8274	.6787	1.4733	50	
	20	.5640	.8258	.6830	1.4641	40	
	30	.5664	.8241	.6873	1.4550	30	
	40	.5688	.8225	.6916	1.4460	20	
	50	.5712	.8208	.6959	1.4370	10	
35	00	.5736	.8192	.7002	1.4281	00	55
	10	.5760	.8175	.7046	1.4193	50	
	20	.5783	.8158	.7089	1.4106	40	
	30	.5807	.8141	.7133	1.4019	30	
	40	.5831	.8124	.7177	1.3934	20	
	50	.5854	.8107	.7221	1.3848	10	
		COS	SIN	COT	TAN	′ ANGLE	°

ANGLE °	′	SIN	COS	TAN	COT		
36	00	.5878	.8090	.7265	1.3764	00	54
	10	.5901	.8073	.7310	1.3680	50	
	20	.5925	.8056	.7355	1.3597	40	
	30	.5948	.8039	.7400	1.3514	30	
	40	.5972	.8021	.7445	1.3432	20	
	50	.5995	.8004	.7490	1.3351	10	
37	00	.6018	.7986	.7536	1.3270	00	53
	10	.6041	.7969	.7581	1.3190	50	
	20	.6065	.7951	.7627	1.3111	40	
	30	.6088	.7934	.7673	1.3032	30	
	40	.6111	.7916	.7720	1.2954	20	
	50	.6134	.7898	.7766	1.2876	10	
38	00	.6157	.7880	.7813	1.2799	00	52
	10	.6180	.7862	.7860	1.2723	50	
	20	.6202	.7844	.7907	1.2647	40	
	30	.6225	.7826	.7954	1.2572	30	
	40	.6248	.7808	.8002	1.2497	20	
	50	.6271	.7790	.8050	1.2423	10	
39	00	.6293	.7771	.8098	1.2349	00	51
	10	.6316	.7753	.8146	1.2276	50	
	20	.6338	.7735	.8195	1.2203	40	
	30	.6361	.7716	.8243	1.2131	30	
	40	.6383	.7698	.8292	1.2059	20	
	50	.6406	.7679	.8342	1.1988	10	
40	00	.6428	.7660	.8391	1.1918	00	50
	10	.6450	.7642	.8441	1.1847	50	
	20	.6472	.7623	.8491	1.1778	40	
	30	.6494	.7604	.8541	1.1708	30	
	40	.6517	.7585	.8591	1.1640	20	
	50	.6539	.7566	.8642	1.1571	10	
41	00	.6561	.7547	.8693	1.1504	00	49
	10	.6583	.7528	.8744	1.1436	50	
	20	.6604	.7509	.8796	1.1369	40	
	30	.6626	.7490	.8847	1.1303	30	
	40	.6648	.7470	.8899	1.1237	20	
	50	.6670	.7451	.8952	1.1171	10	
		COS	SIN	COT	TAN	ANGLE ′	°

ANGLE °	′	SIN	COS	TAN	COT		
42	00	.6691	.7431	.9004	1.1106	00	48
	10	.6713	.7412	.9057	1.1041	50	
	20	.6734	.7392	.9110	1.0977	40	
	30	.6756	.7373	.9163	1.0913	30	
	40	.6777	.7353	.9217	1.0850	20	
	50	.6799	.7333	.9271	1.0786	10	
43	00	.6820	.7314	.9325	1.0724	00	47
	10	.6841	.7294	.9380	1.0661	50	
	20	.6862	.7274	.9435	1.0599	40	
	30	.6884	.7254	.9490	1.0538	30	
	40	.6905	.7234	.9545	1.0477	20	
	50	.6926	.7214	.9601	1.0416	10	
44	00	.6947	.7193	.9657	1.0355	00	46
	10	.6967	.7173	.9713	1.0295	50	
	20	.6988	.7153	.9770	1.0235	40	
	30	.7009	.7133	.9827	1.0176	30	
	40	.7030	.7112	.9884	1.0117	20	
	50	.7050	.7092	.9942	1.0058	10	
45	00	.7071	.7071	1.0000	1.0000	00	45
		COS	SIN	COT	TAN	′ ANGLE	°

27. Functions of Angles Greater than 90°

If the radius vector OA of Figure 6 is rotated from an initial position in the horizontal axis X-X′, it generates the angle XOA which is less than 90°. The trigonometric functions of this angle are those that have been discussed above, i.e., sin XOA = AM/OA, cos XOA = OM/OA, etc. If, however, we now rotate the radius to a new position OA′ we find that a right triangle cannot be constructed on the generated angle XOA′ since it is greater than 90°. It will be observed, however, that this

angle is equal to 180° minus the angle A′OX′ and it may be demonstrated that its trigonometric functions are the same as those of the latter angle except for certain differences in sign. Similarly the angle XOA″ is equal to 180° plus the angle X′OA″ and angle XOA‴ is equal to 360° minus angle A‴ OX; in both cases the functions of the larger angle are the same as those of the associated acute angle with signs changed when necessary.

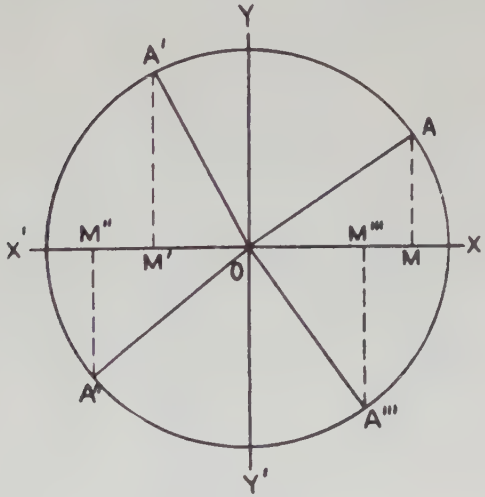


Figure 6

The proper signs in each case are determined by considering all values measured upward from the horizontal axis X—X′ as positive and value measured downward as negative; also, all values measured to the right from the vertical axis Y—Y′ as positive and to the left negative. The radius or moving line describing the angle is always positive. Then, for example, the sine of the angle XOA′

which is equal in magnitude to the sine of angle X′OA′ or A′M′/OA′, is found to have a positive sign since A′M′ is positive, being measured upward from X—X′, and OA′ is always positive, i.e., sin XOA′ = + A′M′/OA′. The cosine of angle XOA′, on the other hand, is negative since the cosine of angle X′OA′ is equal to M′O/OA′ and M′O is negative, being measured to the left of Y—Y′. In the same way we will find that the tangent of angle XOA‴, for example, is negative since it is equal to A‴M‴, which is negative, divided by OM‴, which is positive.

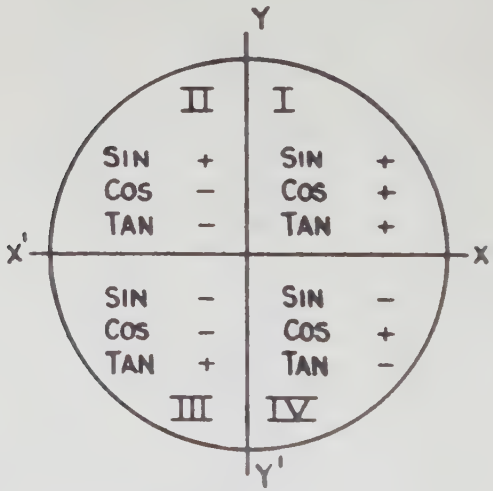


Figure 7

Figure 7 shows the signs of the three primary functions in each of the four quarters or **quadrants** of the circle diagram. This figure may be used as an aid in determining the functions of angles greater than 90° from the tables. If for example, we wish to

know the tangent of 148° we observe that this is in the second quadrant(being between 90° and 180°). From Figure 7 we note that the tangent of an angle in the second quadrant is negative and in the table we look up the tangent of 180—148 or 32° which we find to be .6249. Then we have for the value sought, tangent 148° = —.6249. Similarly, to find the sine of 284°, we note that this is in the fourth quadrant and the sine will therefore be negative; from the table we take the sine of 360—284 or 76°, giving as our result sine 284° = —.9703.

28. Trigonometric Formulae

In solving equations containing trigonometric functions, it is sometimes desirable to express one or more functions in terms of a different angle or a different function. A few formulae showing certain relations between various functions, which, while not proven, may be verified by the substitution of numerical values, are given below:

$$\sin A = \cos (90^\circ - A) = -\sin (-A)$$

$$= \frac{\tan A}{\sqrt{1 + \tan^2 A}}$$

$$\cos A = \sin (90^\circ \pm A) = \cos (-A)$$

$$= \frac{1}{\sqrt{1 + \tan^2 A}}$$

$$\tan A = \cot (90^\circ - A) = -\cot (90^\circ + A)$$

$$= -\tan (-A) = \frac{\sin A}{\cos A}$$

$$\sin^2 A - \cos^2 A = \sec^2 A - \tan^2 A = 1$$

$$1 + \tan^2 A = \frac{1}{\cos^2 A} \qquad 1 + \cot^2 A = \frac{1}{\sin^2 A}$$

$\sin 2A = 2\sin A \cos A$. $\cos 2A = \cos^2 A - \sin^2 A$.

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad \tan \frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

$$\sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} \qquad \cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}$$

$\sin^2 A = \frac{1}{2} (1 - \cos 2A)$. $\cos^2 A = \frac{1}{2} (1 + \cos 2A)$

$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$.

$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$.

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$\sin A \sin B = \frac{1}{2} \cos (A - B) - \frac{1}{2} \cos (A + B)$

$\cos A \cos B = \frac{1}{2} \cos (A + B) + \frac{1}{2} \cos (A - B)$

$\sin A \cos B = \frac{1}{2} \sin (A + B) + \frac{1}{2} \sin (A - B)$

$\cos A \sin B = \frac{1}{2} \sin (A + B) - \frac{1}{2} \sin (A - B)$

$\sin A + \sin B = 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B)$

$\sin A - \sin B = 2 \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$

$\cos A + \cos B = 2 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B)$

$\cos A - \cos B = 2 \sin \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$

29. Examples

1. Determine with the aid of the table the values of the following trigonometric functions:

(a) sin 42°; (b) sin 138°; (c) tan 17°30′

(d) cos 112°20′; (e) sin 252°15′ (f) tan —120°

2. In the triangle of Fig. 4, assume that A = 24.6 and B = 14.1 Find C and angles a and b.

3. In the triangle of Figure 4, assume that b = 33°30′ and A = 42. Find B, C, and angle a.

4. Find A and B in the triangle of Figure 4, if C= 270 and angle a = 70°45′.

5. Using two different arbitrarily selected values for A, show that 2 sin A cos A = sin 2A; $\sin^2 A - \cos^2 A = 1$; and $\cos^2 A - \sin^2 A = \cos 2A$

   Note: $\sin^2 A$ or $\cos^2 A$ means simply the square of sinA or cosA respectively.

6. An airplane flying between cities A and B is prevented from taking a straight line course by a storm. Instead, starting from A, the pilot flies for 184 miles in a straight line which makes an angle of 22° with the straight line between A and B and then turns and flies 268 miles in a straight line to B. What is the distance between A and B ?

30. Vector Quantities

All of the quantities that we have thus far dealt with have been such that they could be adequately expressed simply by a number giving their magnitude. There are, however, certain kinds of quantities which cannot be completely defined by stating their magnitude alone. To define a force, for instance, as having a value of 10 pounds does not mean much unless we also know the direction in which the force is acting.

Figure 8

Such a quantity can be most effectively illustrated by means of an arrow the length of which is scaled to give the magnitude of the force. Then the direction of the force or the line in which it is acting is shown by the position of the arrow. Thus, to represent a force of say, 50 pounds, acting in a vertical direction, we use the arrow of Figure 8, the vertical position of which shows the direction of the force and the length of which is suitably scaled to represent the magnitude, 50. In addition, the

head of the arrow indicates the sense of the force, i.e., whether it is acting upward or downward in the line of the arrow. Such a representation is called a **vector** and quantities which need to be represented in this manner are called **vector quantities.** In addition to force, such quantities include velocity, acceleration, etc. In the study of electricity, it will be found that vectors may also be used very effectively to represent alternating currents and voltages and electrical impedance.

There are naturally many occasions where it becomes necessary to perform mathematical operations with vector quantities and it is therefore desirable that we develop a method of treating such quantities in a manner more or less similar to that which we have previously used in dealing with ordinary algebraic quantities. Addition, subtraction and certain other operations may be performed with vector quantities by graphical methods but as analytical methods are much less cumbersome, we will consider here only the latter.

31. Vector Notation

Any vector, such as OP of Figure 9, may be considered as being made up of two components at right angles to each other and forming the legs of a right triangle having the vector as its hypotenuse. Thus, in Figure 9 we may consider OP as made up of the two components, R and X, where since any number of right triangles could be constructed with OP as hypotenuse, we have arbitrarily placed R in a horizontal position and X in a vertical position.

From simple geometry, then, the **magnitude** of the vector OP is

$$Z = \sqrt{R^2 + X^2} \quad *$$

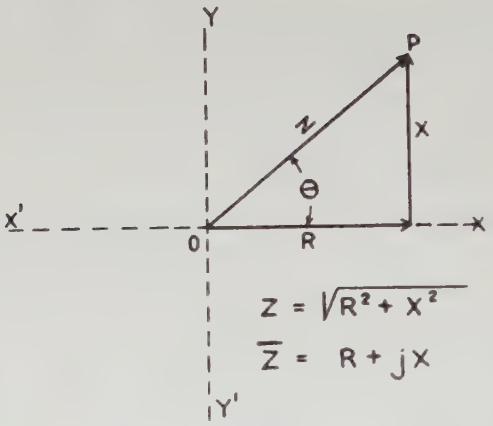


Figure 9

This method of representing a vector, however, does not lend itself readily to mathematical calculations and it is more convenient to express the vector in the form,

$$OP = \overline{Z} = R + jX.$$

This means that the vector is made up of the two components, R and X, the first of which is in a horizontal line and the second in a vertical line, or separated by an angle of 90° from the first.

The latter fact is indicated by the mathematical symbol "j", the meaning of which is a **rotation of 90° in a counter-clockwise direction.** Thus if we

*The selection of the letters, Z, R and X, to represent the vector and its components is here entirely arbitrary; any other three letters would have served equally well.

multiply a vector lying in a horizontal position, such as A of Figure 10, by j, the vector is rotated 90° in a counter-clockwise direction to the new

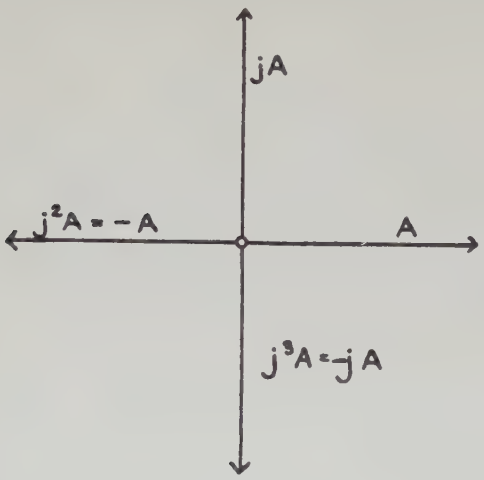


Figure 10

position indicated by jA; and if we again multiply by j, the vector is rotated through another 90° to a position 180° from its original position. Hence we see that

$$j\ (jA) = j^2A = -A = (-1)A$$

from which $j^2 = -1$ and $j = \sqrt{-1}$

Likewise if we multiply again by j we have a positive (counter-clockwise) rotation of 270° from the original position or a negative (clockwise) rotation of 90°, thus—

$$j\ (j^2A) = j^3A = j(-A) = -jA.$$

In using the operator, j, it is necessary to keep in mind its mathematical significance and to remember that "j terms" can be combined algebraically only with other "j terms" and that "j terms" can be combined with "non-j terms" only by the geometric relationship

$$R + jX = \sqrt{R^2 + X^2}$$

In should be noted, however, that $\sqrt{R^2 + X^2}$ tells us only the magnitude of the vector while the expression, $R + jX$, tells us not only the magnitude but also the angular position of the vector, since only one position of the hypotenuse is possible for a right triangle having legs R and X in a horizontal and a vertical position respectively.

Now referring again to Figure 9, it will be seen that R is equal to Z cos Θ where Θ is the angle that the vector $\overline{Z}$, makes with the horizontal axis, $X - X'$; also, that X is equal to Z sin Θ. Hence we may write

$$OP = \overline{Z} = R + jX = Z \cos \Theta + jZ \sin \Theta =$$

$$Z (\cos \Theta + j \sin \Theta)$$

In this expression the vector is completely described in terms of its magnitude, Z, and its angle, Θ. Further, it can be shown by the use of infinite series that $(\cos \Theta + j \sin \Theta)$ may be written in an exponential form, thus—

$$(\cos \Theta + j \sin \Theta) = e^{j\Theta}$$

so that we now have

$$OP = \overline{Z} = R + j X = Ze^{j\Theta}$$

This latter form, namely, $Ze^{j\theta}$ is usually written for convenience simply $Z \angle \theta$ (Z at an angle θ) but it is well to keep in mind the fact that θ actually occurs as an exponent.

32. Operations on Vectors

From our knowledge of the properties of exponents as given in Article 20, we will observe that the mathematical operations of multiplication, division, raising to a power or extracting a root may be conveniently performed with vectors expressed in the $Z\angle\theta$ or $Ze^{j\theta}$ form. Thus, for instance, if we wish to multiply together the vectors $Z_1 \angle \theta_1$ and $Z_2 \angle \theta_2$, we have

$$Z_1\angle\theta_1 \times Z_2\angle\theta_2 = Z_1 e^{j\theta_1} \times Z_2 e^{j\theta_2} =$$
$$Z_1 Z_2 e^{j(\theta_1 + \theta_2)} = Z_1 Z_2 \angle \theta_1 + \theta_2.$$

In adding or subtracting vectors, on the other hand, it is more convenient to express the vectors in the $R + jX$ form, in which case it is only necessary to combine separately the "j" and "non-j" terms. Thus—

$$(R_1 + jX_1) + (R_2 + jX_2) = (R_1 + R_2) + j\,(X_1 + X_2)$$

Vectors expressed in the $R + jX$ form may be readily converted to the $Z\angle\theta$ form by means of the following (See Figure 9):

$$\theta = \tan^{-1} \frac{X}{R}$$

$$Z = \frac{R}{\cos \theta} \text{ or } \frac{X}{\sin \theta}$$

And, as we have already noted, conversion may be made from the $Z/\underline{\Theta}$ form to the $R + j X$ form by use of the simple trigonometric relationships.

$$R = Z \cos \Theta$$

and $$X = Z \sin \Theta.$$

The most convenient methods of performing the common mathematical operations on vector quantities may be summarized as follows:

(a) **Addition of Vectors**

Express the vectors in the $R + j X$ form.

Add separately the "j" and "non-j" terms, thus—

$$(R_1 + jX_1) + (R_2 + jX_2) = (R_1 + R_2) + j (X_1 + X_2)$$

(b) **Subtraction of Vectors**

Express the vectors in the $R + j X$ form.

Subtract separately the "j" and "non-j" terms, thus—

$$(R_1 + j X_1) - (R_2 + j X_2) = (R_1 - R_2) + j (X_1 - X_2)$$

(c) **Multiplication of Vectors**

Express the vectors in the $Z/\underline{\Theta}$ form.

Multiply the magnitudes and add the angles, thus—

$$Z_1 / \underline{\Theta_1} \times Z_2 / \underline{\Theta_2} = Z_1 Z_2 / \underline{\Theta_1 + \Theta_2}$$

(d) **Division of Vectors**

Express the vectors in the $Z / \underline{\Theta}$ form.

Divide the magnitudes and subtract the angles, thus—

$$\frac{Z_1 \angle \Theta_1}{Z_2 \angle \Theta_2} = \frac{Z_1}{Z_2} \angle \Theta_1 - \Theta_2$$

(e) **Power of a Vector**

Express the vector in the $Z \angle \Theta$ form.

Raise the magnitude to the power indicated by the exponent and multiply the angle by the exponent, thus—

$$(Z \angle \Theta)^n = Z^n \angle n\Theta$$

(f) **Root of a Vector**

Express the vector in the $Z \angle \Theta$ form.

Take the root of the magnitude indicated by the index and divide the angle by the index, thus—

$$\sqrt[n]{Z \angle \Theta} = \sqrt[n]{Z} \angle \Theta/n$$

33. Calculations with Vector Quantities

The manner in which the notation just described can be employed in calculations involving vector quantities may be best shown by means of practical examples. As an instance, consider the problem of finding the net force acting upon a body which has applied to it two separate forces acting in two differing directions.

The situation is illustrated in Figure 11 where forces of 40 pounds and 60 pounds are shown acting on a body A at respective angles of 25° and 70° with the arbitrarily selected reference line X — X′

passing through the common point of application of the forces. Our problem is to find the magnitude and direction of the single force, X, which would have exactly the same effect on the body A as the two forces given.

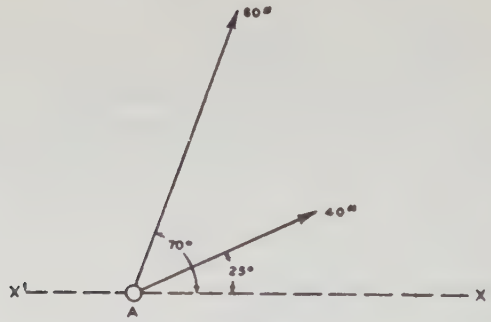


Figure 11

It is reasonably clear that X is simply the vector sum of the two separate vectors. To obtain this sum, we must first express the vectors in the R + jX form. We have for the smaller force—

$$40 \underline{/\ 25^\circ} = 40 \cos 25^\circ + j40 \sin 25^\circ$$
$$= 40 \times .9063 + j40 \times .4226$$
$$= 36.25 + j\ 16.90$$

and, for the larger vector—

$$60 \underline{/\ 70^\circ} = 60 \cos 70^\circ + j\ 60 \sin 70^\circ$$
$$= 60 \times .3420 + j\ 60 \times .9397$$
$$= 20.52 + j56.38$$

Then adding, we have for X

$$X = 36.25 + j\ 16.90 + 20.52 + j56.38$$
$$= 36.25 + 20.52 + j\ (16.90 + 56.38)$$
$$= 56.77 + j\ 73.28$$

To know the magnitude and angle of the net force X, we must express this result in the $Z \angle \Theta$ form. To find Θ, we make use of the fact that

$$\Theta = \tan^{-1} \frac{X}{R} = \tan^{-1} \frac{73.28}{56.77} = \tan^{-1} 1.291 = 52°15'$$

$$\text{and } Z = \frac{R}{\cos \Theta} = \frac{56.77}{\cos 52°15'} = \frac{56.77}{.612} = 92.7$$

Our sum vector is then

$$X = 92.7 \angle 52°15'$$

which means that the net force acting on A is 92.7 pounds acting in a line which makes a positive angle of 52°15′ with X — X′.

As a further example, let us consider a problem frequently encountered in alternating current calculations—to find the unknown vector which equals the product of two known vectors divided by their sum. The equation is

$$Z_o \angle \Theta_o = \frac{Z_1 \angle \Theta_1 \times Z_2 \angle \Theta_2}{Z_1 \angle \Theta_1 + Z_2 \angle \Theta_2}$$

where $Z_1 \angle \Theta_1$ and $Z_2 \angle \Theta_2$ are known and $Z_o \angle \Theta_o$ is unknown. Assuming values of $10 \angle 40°$ and $8 \angle -16°$ for the values of $Z_1 \angle \Theta_1$ and $Z_2 \angle \Theta_2$ respectively, we have

$$Z_o \angle \Theta_o = \frac{10 \angle 40° \times 8 \angle -16°}{10 \angle 40° + 8 \angle -16°}$$

We can perform the multiplication indicated in the numerator with the vectors expressed in their

present form but to effect the addition indicated in the denominator, the two vectors must be converted to the R + jX form. Doing this first, we have

$$Z_o \underline{/\ \theta_o} = \frac{10 \underline{/\ 40^\circ} \times 8 \underline{/\ -16^\circ}}{10 \cos 40^\circ + j\ 10 \sin 40^\circ + 8 \cos 16^\circ - j\ 8 \sin 16^\circ}$$

$$= \frac{10 \underline{/\ 40^\circ} \times 8 \underline{/\ -16^\circ}}{7.66 + j6.43 + 7.69 - j2.21}$$

Then, performing the indicated multiplication and additions,

$$Z_o \underline{/\ \theta_o} = \frac{10 \times 8 \underline{/\ 40^\circ - 16^\circ}}{7.66 + 7.69 + j(6.43 - 2.21)}$$

$$= \frac{80 \underline{/\ 24^\circ}}{15.35 + j\ 4.22}$$

To simplify the fraction, we must convert the denominator to the $Z \underline{/\ \theta}$ form. The angle is

$$\theta = \tan^{-1} \frac{4.22}{15.35} = \tan^{-1} .275 = 15^\circ 23'$$

and $$Z = \frac{15.35}{\cos 15^\circ 23'} = \frac{15.35}{.9641} = 15.92$$

Hence $$Z_o \underline{/\ \theta_o} = \frac{80 \underline{/\ 24^\circ}}{15.92 \underline{/\ 15^\circ 23'}}$$

From which, dividing, we get

$$Z_o / \underline{\Theta_o} = \frac{80}{15.92} / \underline{24^\circ - 15^\circ 23'}$$

$$= 5.02 / \underline{8^\circ 37'}$$

34. Examples

1. Express the following in the R + j X form:

   (a) $892/\underline{16^\circ 30'}$; (b) $240/\underline{-60^\circ}$; (c) $.48/\underline{200^\circ}$

2. Express the following in the $Z / \underline{\Theta}$ form:

   (a) 24 + j80; (b) 295 — j100; (c) 10 + j0;

   (d) 0 — j50 (e) — 82 + j10; (f) —6 — j9.

3. Combine the following three forces to find their net resultant:

   (1) 75# acting at a positive angle with the horizontal of 10°

   (2) 100# acting at a positive angle with the horizontal of 90°

   (3) 150# acting at a negative angle with the horizontal of 30°

4. Determine the product of 8 +j12 and 7 --j11.

5. What are the values of the following expressed in the R + jX form:

   (a) $(.42 / \underline{27^\circ})^2$

   (b) $\sqrt{8.9 / \underline{-82^\circ}}$

   (c) $\dfrac{24 / \underline{0^\circ}}{16 / \underline{-10^\circ}}$

P9-EBO-372